Convergence and qualitative properties of modified explicit schemes for BSDEs with polynomial growth

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Abstract

These notes contain the verifications that that our abstract assumptions are satisfied for the main three examples of taming considered in the paper.

This guarantees that for all the examples treated in section 6, we are in the framework we developped and the convergence is proven. It also allows to track down the final convergence rate.

Contents

1 The assumptions on the drivers

1.1 Assumptions on the driver of the BSDE

(Gr) There exist $m \in \mathbb{N}^*$ and constants $K_t, K_y, K_z \geq 0$ such that, for all $(t, y, z) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n\times d}$,

$$|f(t, y, z)| \leq K_t + K_y|y|^m + K_z|z| \ . \tag{1.1}$$

That is, f has polynomial growth in y of degree m and linear growth in z.

(Mon) There exist a constant $M_y \in \mathbb{R}$ such that for all t, y, y', z,

$$\langle y' - y, f(t, y', z) - f(t, y, z)\rangle \leq M_y|y' - y|^2 \ . \tag{1.2}$$

That is, f is monotone ("decreasing") in the variable y, with monotonicity constant M_y, which can be, but is not necessarily, strictly negative.

(Reg) There exist constants $L_t, L_z \geq 0$ such that for all t, t', y, z',

$$|f(t', y, z') - f(t, y, z)| \leq L_t|t' - t|^{\frac{1}{2}} + L_z|z' - z| \ . \tag{1.3}$$

That is, f is $\frac{1}{2}$-Hölder in time and Lipschitz in z.

(RegY) There exists a constant $L_y \geq 0$ such that for all t, y, y', z,

$$|f(t, y', z) - f(t, y, z)| \leq L_y\big(1 + |y'|^{m-1} + |y|^{m-1}\big)|y' - y| \ . \tag{1.4}$$

That is, f is locally Lipschitz in y with local Lipschitz constant growing polynomially with degree $m - 1$, with the m from **(Gr)** .

(MonGr) There exist constants $\bar{M}_t, \bar{M}_z \geq 0$ and $\bar{M}_y \in \mathbb{R}$ such that for all t, y, z,

$$\langle y, f(t, y, z)\rangle \leq \bar{M}_t + \bar{M}_y|y|^2 + \bar{M}_z|z|^2 \ . \tag{1.5}$$

Remark 1.1. *If f satisfies **(Mon)** and **(Gr)** , then for all t, y, z and for any $\alpha > 0$, we have*

$$\begin{aligned}
\langle y, f(t, y, z)\rangle &= \langle y - 0, f(t, y, z) - f(t, 0, z)\rangle + \langle y, f(t, 0, z)\rangle \\
&\leq M_y|y - 0|^2 + |y|\,\big(K_t + K_z|z|\big) \\
&\leq (M_y + \alpha)|y|^2 + \frac{K_t^2}{2\alpha} + \frac{K_z^2}{2\alpha}|z|^2 \ .
\end{aligned}$$

*Hence we can take $\bar{M}_t = \frac{K_t^2}{2\alpha}$ and $\bar{M}_z = \frac{K_z^2}{2\alpha}|z|^2$ arbitrarily small, while taking $\bar{M}_y = M_y + \alpha$. We also note that by combining **(Mon)** and **(Reg)** we obtain the general estimate*

$$\begin{aligned}
\langle y' - y, f(t, y', z') - f(t, y, z)\rangle &= \langle y' - y, f(t, y', z') - f(t, y, z')\rangle \\
&\qquad + \langle y' - y, f(t, y, z') - f(t, y, z)\rangle \\
&\leq M_y|y' - y|^2 + |y' - y|\, L_z|z' - z| \\
&\leq (M_y + \alpha)|y' - y|^2 + \frac{L_z^2}{4\alpha}|z' - z|^2 \ .
\end{aligned}$$

*Compare with Remark 2.1 in [**?**].*

1.2 Assumptions on the tamed driver of the scheme

(TGr) There exist K_t^h, K_y^h and $K_z^h \geq 0$ such that, for all $(t,y,z) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^{n\times d}$,

$$|f^h(t,y,z)| \leq K_t^h + K_y^h|y| + K_z^h|z| \ .$$

The constants K_t^h, K_y^h and K_z^h may depend on h but in such a way that $(K_t^h)^2h$, $(K_y^h)^2h$ and K_z^h are bounded in h.

(TMonGr) There exist $\bar{M}_t^h, \bar{M}_z^h \geq 0$ and $\bar{M}_y^h \in \mathbb{R}$ such that, for all $(t,y,z) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^{n\times d}$,

$$\langle y, f(t,y,z)\rangle \leq \bar{M}_t^h + \bar{M}_y^h|y|^2 + \bar{M}_z^h|z|^2 \ .$$

The constants $\bar{M}_t^h, \bar{M}_y^h, \bar{M}_z^h$ may depend on h, but are bounded in h.

(TReg) There exist $L_t^h, L_z^h \geq 0$ such that, for all t, t', y, z, z',

$$|f^h(t',y,z') - f^h(t,y,z)| \leq L_t^h|t'-t|^{\frac{1}{2}} + L_z^h|z'-z| \ .$$

L_t^h and L_z^h may depend on h, but in a bounded way.

(TRegY) There exist $L_y^h \geq 0$ and a function $\mathcal{R}^{(\mathrm{regY})}$ such that $|\mathcal{R}^{(\mathrm{regY})}|$ satisfies **(TCvg)** such that for all t, y, y', z,

$$|f^h(t,y',z) - f^h(t,y,z)| \leq L_y^h|y'-y| + \mathcal{R}^{(\mathrm{regY})}(t,y',y,z).$$

L_y^h may depend on h, but in such a way that $(L_y^h)^2h$ is bounded in h.

(TMon) There exists $M_y^h \in \mathbb{R}$ and a function $\mathcal{R}^{(\mathrm{mon})}$ satisfying **(TCvg)** such that for all t, y, y', z,

$$\left\langle y'-y, f^h(t,y',z) - f^h(t,y,z)\right\rangle \leq M_y^h|y'-y|^2 + \mathcal{R}^{(\mathrm{mon})}(t,y',y,z) \ .$$

M_y^h may depend on h, but in a bounded way.

We need to ensure that $f^h \to f$ as $h \to 0$. This is in some sense a *consistency condition*, ensuring that the output of the scheme converges to the solution to the correct BSDE, and not a BSDE with a different driver. We introduce for this $R^h = f - f^h$. Also, we need the remainders $\mathcal{R}^{(\mathrm{regY})}$ and $\mathcal{R}^{(\mathrm{mon})}$ to vanish sufficiently fast, so as not to prevent convergence of the scheme. The following assumptions guarantees that R^h and $\mathcal{R}$ converge to 0, where $\mathcal{R}$ is any of the remainders $\mathcal{R}^{(\mathrm{regY})}$ and $\mathcal{R}^{(\mathrm{mon})}$.

(TCvg) One of the following holds.

1. There exist constants $C \geq 0$, $p, q \geq 1$ and $\alpha > 0$ such that for any y', y, z

$$|R^h(y,z)| \leq C\,\left(1 + |y|^q + |z|^p\right)\, h^\alpha$$
$$\mathcal{R}(t,y',y,z) \leq C\big(1 + |y'|^q + |y|^q + |z|^p\big)h^\alpha.$$

2. There exist constants $C \geq 0$, $p, q \geq 1$, $r_0 > 0$ and $\beta > 0$ such that, with $r(h) = r_0 h^{-\beta}$, for any y', y, z

$$|R^h(y,z)| \leq C\,\left(1 + |y|^q + |z|^q\right)1_{\{|f(y,z)|>r(h)\}}$$
$$\mathcal{R}(t,y',y,z) \leq C\big(1 + |y'|^q + |y|^q + |z|^p\big)\ 1_{\{|f(t,y',z)|>r(h) \text{ or } |f(t,y,z)|>r(h)\}}.$$

3. There exist constants $C \geq 0$, $p, q \geq 1$, $r_0 > 0$ and $\gamma > 0$ such that, with $r(h) = r_0 h^{-\gamma}$, for any y', y, z

$$|R^h(y,z)| \leq C\,\left(1 + |y|^q + |z|^p\right)1_{\{|y|>r(h)\}}$$
$$\mathcal{R}(t,y',y,z) \leq C\big(1 + |y'|^q + |y|^q + |z|^p\big)\ 1_{\{|y'|>r(h) \text{ or } |y|>r(h)\}}.$$

2 Verification that the usual tamed drivers fit in our framework

2.1 Verifications for the multiplicative taming

Consider a radius $r(h) = r_0 h^{-\alpha}$. The multiplicative taming is given by

$$f^h(t,y,z) = \chi^h(y) f(t,y,z), \quad \text{where} \quad \chi^h(y) = \frac{1}{1 + F(y) r(h)^{-1}}.$$

Several choices are possible for the function F. We consider the four following ones.

(a) $F(y) = |f(0,y,0)|$.

(b) $F(y) = \frac{|f(0,y,0) - f(0,0,0)|}{|y|} 1_{\{y \neq 0\}}$.

(c) $F(y) = |y|^m$.

(d) $F(y) = |y|^{m-1}$.

Before starting, let us notice that we have $0 \leq \chi^h(y) \leq 1$ and also that

$$1 - \chi^h(y) = \frac{F(y) r(h)^{-1}}{1 + F(y) r(h)^{-1}} = \chi^h(y) F(y) r(h)^{-1} \leq F(y) r(h)^{-1}.$$

2.1.1 Verification of (TGr)

Using **(TReg)** if there is a t or z dependence, we have first

$$\begin{aligned} |f^h(t,y,z)| &\leq \chi^h(y)|f(t,y,z) - f(0,y,0)| + \chi^h(y)|f(0,y,0)| \\ &\leq 1 \times \left(L_t T^{\frac{1}{2}} + L_z |z|\right) + \chi^h(y)|f(0,y,0)|. \end{aligned}$$

Here we need to distinguish the cases.

Case (a).

$$\chi^h(y)|f(0,y,0)| = \frac{|f(0,y,0)|}{1 + F(y) r(h)^{-1}} = \frac{|f(0,y,0)|}{1 + |f(0,y,0)| r(h)^{-1}} \leq r(h).$$

So we have in the end

$$|f^h(t,y,z)| \leq \left(L_t T^{\frac{1}{2}} + r(h)\right) + 0 + L_z |z|.$$

We take $K_t^h = L_t T^{\frac{1}{2}} + r(h)$, $K_y^h = 0$ and $K_z^h = L_z$. Recalling that $r(h) = r_0 h^{-\alpha}$, the condition $(K_t^h)^2 h$ bounded as $h \to 0$ is equivalent to $\alpha \leq \frac{1}{2}$.

Case (b).

$$\begin{aligned} \chi^h(y)|f(0,y,0)| &= \chi^h(y)|f(0,y,0) - f(0,0,0)| + \chi^h(y)|f(0,0,0)| \\ &\leq \frac{|f(0,y,0) - f(0,0,0)|}{1 + \frac{|f(0,y,0) - f(0,0,0)|}{|y|} 1_{\{y \neq 0\}} r(h)^{-1}} + |f(0,0,0)| \\ &\leq |y| r(h) + |f(0,0,0)|. \end{aligned}$$

So we have in the end

$$|f^h(t,y,z)| \leq \left(L_t T^{\frac{1}{2}} + |f(0,0,0)|\right) + r(h)|y| + L_z |z|.$$

We take $K_t^h = L_t T^{\frac{1}{2}} + |f(0,0,0)|$, $K_y^h = r(h)$ and $K_z^h = L_z$. This time, the condition to check is that $(K_y^h)^2 h$ is bounded as $h \to 0$, and this is again equivalent to $\alpha \leq \frac{1}{2}$.

Case (c). Using **(Gr)** we see that

$$\chi^h(y)|f(0,y,0)| = \frac{|f(0,y,0)|}{1+|y|^m r(h)^{-1}} \leq \frac{K_t + K_y|y|^m}{1+|y|^m r(h)^{-1}} \leq K_t + K_y r(h).$$

So we have in the end

$$|f^h(t,y,z)| \leq \big(L_t T^{\frac{1}{2}} + K_t + K_y r(h)\big) + 0 + L_z|z|.$$

We take $K_t^h = L_t T^{\frac{1}{2}} + K_t + K_y r(h)$, $K_y^h = 0$ and $K_z^h = L_z$. Again, $(K_t^h)^2 h$ bounded as $h \to 0$ is equivalent to $\alpha \leq \frac{1}{2}$.

Case (d). Using **(Gr)** again we see that

$$\chi^h(y)|f(0,y,0)| = \frac{|f(0,y,0)|}{1+|y|^{m-1} r(h)^{-1}} \leq \frac{K_t + K_y|y|^m}{1+|y|^{m-1} r(h)^{-1}} \leq K_t + K_y r(h)|y|.$$

So we have in the end

$$|f^h(t,y,z)| \leq \big(L_t T^{\frac{1}{2}} + K_t\big) + K_y r(h)|y| + L_z|z|.$$

We take $K_t^h = L_t T^{\frac{1}{2}} + K_t$, $K_y^h = K_y r(h)$ and $K_z^h = L_z$. Again, $(K_y^h)^2 h$ bounded as $h \to 0$ is equivalent to $\alpha \leq \frac{1}{2}$.

We have therefore verified **(TGr)** in the four cases.

2.1.2 Verification of (TMonGr)

We use the fact that f satisfies **(MonGr)** , as well as $\chi^h(y) \in [0,1]$, to write

$$\begin{aligned}
\big\langle y, f^h(t,y,z)\big\rangle &= \chi^h(y)\,\langle y, f(t,y,z)\rangle \\
&\leq \chi^h(y)\Big(\bar{M}_t + \bar{M}_y|y|^2 + \bar{M}_z|z|^2\Big) \\
&\leq \bar{M}_t + \chi^h(y)\bar{M}_y|y|^2 + \bar{M}_z|z|^2 \\
&\leq \bar{M}_t^h + \bar{M}_y^h|y|^2 + \bar{M}_z^h|z|^2
\end{aligned}$$

where $\bar{M}_t^h = \bar{M}_t$, $\bar{M}_z^h = \bar{M}_z$ and $\bar{M}_y^h = \max(0, \bar{M}_y)$. These constants do not depend on h, so they work for **(TMonGr)** (they are bounded as $h \to 0$).

2.1.3 Verification of (TReg)

Here again, we use the fact that f satisfies **(Reg)** , as well as $\chi^h(y) \in [0,1]$, to write

$$|f^h(t',y,z') - f^h(t,y,z)| = \chi^h(y)|f(t',y,z') - f(t,y,z)| \leq L_t|t'-t|^{\frac{1}{2}} + L_z|z'-z|.$$

So we take $L_t^h = L_t$ and $L_z^h = L_z$ and **(TReg)** is satisfied.

2.1.4 Verification of (TRegY)

We write

$$\begin{aligned}
f^h(t,y',z) - f^h(t,y,z) &= \chi^h(y')f(t,y',z) - \chi^h(y)f(t,y,z) \\
&= \chi^h(y')\chi^h(y)\big(f(t,y',z) - f(t,y,z)\big) + \mathcal{R}^{(\mathrm{regY})}(t,y,y',z),
\end{aligned}$$

where

$$\mathcal{R}^{(\mathrm{regY})}(t,y,y',z) := \chi^h(y')\big(1-\chi^h(y)\big)f(t,y',z) - \big(1-\chi^h(y')\big)\chi^h(y)f(t,y,z).$$

We first estimate the "good term", that will give the Lipschitz-in-Y regularity for fixed h, and then will estimate the remainder term. Using **(TRegY)** , we have

$$\begin{aligned}|\chi^h(y')\chi^h(y)\big(f(t,y',z)-f(t,y,z)\big)| &\le \chi^h(y')\chi^h(y)L_y\big(1+|y'|^{m-1}+|y|^{m-1}\big)|y'-y| \\ &\le L_y\big(1+\chi^h(y')|y'|^{m-1}+\chi^h(y)|y|^{m-1}\big)|y'-y|.\end{aligned}$$

To estimate the terms $\chi^h(y)|y|^{m-1}$ we need to distinguish cases. We start with the easier cases (c) and (d).

Case (c).

$$\chi^h(y)|y|^{m-1} = \frac{|y|^{m-1}}{1+|y|^{m-1}r(h)^{-1}} \le r(h).$$

So we take $L_y^h := L_y(1+2r(h))$. And $(L_y^h)^2h$ is bounded iff $\alpha \le \frac{1}{2}$.

Case (d).

$$\chi^h(y)|y|^{m-1} = \frac{|y|^{m-1}}{1+|y|^m r(h)^{-1}} \le \begin{cases} r(h) & \text{if } |y|\ge 1 \\ \quad 1 & \text{if } |y| \le 1\end{cases} \le 1+r(h).$$

So we take $L_y^h := L_y(3+2r(h))$. And $(L_y^h)^2h$ is bounded iff $\alpha \le \frac{1}{2}$.

Now, cases (a) and (b) require an extra assumption on f if we want to go on with a general driver f. Namely, we assume that "$|y|^m$ is dominant in f", that is to say : there exist $R \ge 1$ and $k_y \ge 0$ such that for all $y : |y| \ge R$, $|f(0,y,0)| \ge k_y|y|^m$. Notice that in our numerical examples, we worked with the drivers that are polynomials of degree $m = 2$ or 3, and that this assumption is satisfied for any polynomial in y. Therefore, it is proven that for the examples of drivers we used and for the taming applied to them, the resulting drivers f^h fit in our framework.

Case (a).

$$\chi^h(y)|y|^{m-1} = \frac{|y|^{m-1}}{1+|f(0,y,0)|r(h)^{-1}} \le \begin{cases} \dfrac{|y|^{m-1}}{k_y|y|^m}r(h) & \text{if } |y|\ge R\ge 1 \\ R^{m-1} & \text{if } |y|\le R\end{cases} \le R^{m-1}+\frac{1}{k_y}r(h).$$

So we take $L_y^h := L_y\big(1+2R^{m-1}+\frac{2}{k_y}r(h)\big)$. And $(L_y^h)^2h$ is bounded iff $\alpha \le \frac{1}{2}$.

Case (b).

$$\chi^h(y)|y|^{m-1} = \frac{|y|^{m-1}}{1+\frac{|f(0,y,0)-f(0,0,0)|}{|y|}1_{\{y\neq 0\}}r(h)^{-1}} \le \begin{cases} \dfrac{|y|^m}{|f(0,y,0)-f(0,0,0)|}r(h) & \text{if } |y|\ge R' \\ (R')^{m-1} & \text{if } |y|\le R'\end{cases},$$

where $R' \ge R$ is chosen such that for $|y| \ge R'$, $|f(0,y,0)| \ge k_y|y|^m \ge |f(0,0,0)| + \frac{1}{2}k_y|y|^m$. This way, $|f(0,y,0)-f(0,0,0)| \ge |f(0,y,0)|-|f(0,0,0)| \ge \frac{1}{2}k_y|y|^m$. We then have

$$\chi^h(y)|y|^{m-1} \le \begin{cases} \dfrac{2}{k_y}r(h) & \text{if } |y|\ge R' \\ (R')^{m-1} & \text{if } |y|\le R'\end{cases} \le (R')^{m-1}+\frac{2}{k_y}r(h).$$

So we take $L_y^h := L_y\big(1+2(R')^{m-1}+\frac{4}{k_y}r(h)\big)$. And $(L_y^h)^2h$ is bounded iff $\alpha \le \frac{1}{2}$.

In conclusion, the "good term" is indeed bounded from above by $L_y^h|y' - y|$ and $(L_y^h)^2 h$ remain bounded as $h \to 0$. It remains to estimate the remainder term.

We recall before starting that $\chi^h(y) \in [0,1]$ and that

$$1 - \chi^h(y) = \frac{F(y)r(h)^{-1}}{1 + F(y)r(h)^{-1}} = \chi^h(y)F(y)r(h)^{-1} \le F(y)r(h)^{-1}.$$

Then, we estimate

$$\begin{aligned}
|\mathcal{R}^{(\mathrm{regY})}(t,y,y',z)| &= \left|\chi^h(y')\big(1-\chi^h(y)\big)f(t,y',z) - \big(1-\chi^h(y')\big)\chi^h(y)f(t,y,z)\right| \\
&\le \chi^h(y')\big(1-\chi^h(y)\big)\,|f(t,y',z)| + \big(1-\chi^h(y')\big)\chi^h(y)\,|f(t,y,z)| \\
&\le \Big\{1 \times F(y) \times |f(t,y',z)| + F(y') \times 1 \times |f(t,y,z)|\Big\} r(h)^{-1}.
\end{aligned}$$

Now, we use **(Gr)** and **(RegY)** to claim that $F(y) \le C(1+|y|^m)$. For this we separate again the cases.

Case (a) $F(y) = |f(0,y,0)| \le K_t + K_y|y|^m \le C(1+|y|^m)$.

Case (b) $F(y) = \frac{|f(0,y,0)-f(0,0,0)|}{|y|}1_{\{y\neq 0\}} \le L_y(1+|y|^{m-1}) \le C(1+|y|^m)$.

Case (c) $F(y) = |y|^m \le C(1+|y|^m)$.

Case (d) $F(y) = |y|^{m-1} \le C(1+|y|^m)$.

In the above, we have used $|y|^p \le 1 + |y|^q$ for $q \ge p$. Using again **(Gr)** we therefore have

$$\begin{aligned}
|\mathcal{R}^{(\mathrm{regY})}(t,y,y',z)| &\le C\Big\{\big(1+|y|^m\big)\big(1+|y'|^m+|z|\big) + \big(1+|y'|^m\big)\big(1+|y|^m+|z|\big)\Big\} r(h)^{-1} \\
&\le C\Big\{1 + |y|^{2m} + |y'|^{2m} + |z|^2\Big\} h^\alpha.
\end{aligned}$$

To obtain the last estimate we have used several times the inequality $ab \le a^2 + b^2$. Therefore, **(TRegY)** is verified.

2.1.5 Verification of (TMon)

We first write

$$\begin{aligned}
f^h(t,y,z) &= f(t,y,z) + (\chi^h(y) - 1)f(t,y,z) \\
&= f(t,y,z) - F(y)r(h)^{-1}\chi^h(y)f(t,y,z).
\end{aligned}$$

Then, using **(Mon)** for f,

$$\begin{aligned}
\big\langle y'-y, f^h(t,y',z) - f^h(t,y,z)\big\rangle &= \langle y'-y, f(t,y',z) - f(t,y,z)\rangle \\
&\quad - \big\langle y'-y, F(y')\chi^h(y')f(t,y',z) - F(y)\chi^h(y)f(t,y,z)\big\rangle\, r(h)^{-1} \\
&\le M_y\,|y'-y|^2 + \mathcal{R}^{(\mathrm{mon})}(t,y',y,z),
\end{aligned}$$

where

$$\mathcal{R}^{(\mathrm{mon})}(t,y',y,z) = -\big\langle y'-y, F(y')\chi^h(y')f(t,y',z) - F(y)\chi^h(y)f(t,y,z)\big\rangle\, r(h)^{-1}.$$

We now estimate $\mathcal{R}^{(\mathrm{mon})}$ similarly to what we did to verify **(TRegY)** , using $\chi^h(y) \le 1$ and $F(y) \le C(1+|y|^m)$.

$$\begin{aligned}
|\mathcal{R}^{(\mathrm{mon})}(t,y',y,z)| &\le |y'-y|\Big[F(y')|f(t,y',z)| + F(y)|f(t,y,z)|\Big] r(h)^{-1} \\
&\le C\big(|y'|+|y|\big)\Big[\big(1+|y'|^m\big)\big(1+|y'|^m+|z|\big) + \big(1+|y|^m\big)\big(1+|y|^m+|z|\big)\Big] h^\alpha \\
&\le C\Big[1 + |y'|^{4m} + |y|^{4m} + |z|^2\Big] h^\alpha.
\end{aligned}$$

Again, the last inequality resulted from using several Young inequalities and $|y|^p \leq 1 + |y|^q$ for $q \geq p$. Remark in particular that z only appears in a term the form $(|y'| + |y|)(1 + |y'|^m + |y|^m)|z| = ab$ with $b = |z|$.

Therefore, **(TMon)** is verified.

This completes the verification that, when a general driver f is transformed into f^h by a multiplication by a taming factor χ^h, the resulting f^h satisfies **(Gr)** , **(TMonGr)** , **(TReg)** **(TRegY)** and **(TMon)** .

2.1.6 Verification of (TCvg)

$$R^h(t, y, z) = f(t, y, z) - f^h(t, y, z) = f(t, y, z)(1 - \chi^h(y)) = f(t, y, z)\chi^h(y)F(y)r(h)^{-1}.$$

So, using as before $F(y) \leq C(1 + |y|^m)$,

$$\begin{aligned}
\left|R^h(t, y, z)\right| &\leq F(y)|f(t, y, z)|r(h)^{-1} \\
&\leq C(1 + |y|^m)(1 + |y|^m + |z|)h^\alpha \\
&\leq C(1 + |y|^{2m} + |z|^2)h^\alpha.
\end{aligned}$$

Therefore f^h satisfies **(TCvg)** with the criterion 1.

2.2 Verifications for the outer tamings

Consider a radius $r(h) = r_0 h^{-\beta}$. The outer taming is given by

$$f^h(t, y, z) = T^h\big(f(t, y, z)\big),$$

where T^h is essentially a projection on the ball of $\mathbb{R}^n$ of center 0 and radius $r(h)$. Specifically, we can consider only the following two choices.

- The (pure) projection : $T^h(f) = \frac{f}{\max(1, |f| r(h)^{-1})} = \frac{r(h) f}{\max(r(h), |f|)}$.
- A smoothed projection : $T^h(f) = \frac{f}{1 + |f| r(h)^{-1}} = \frac{r(h) f}{r(h) + |f|}$.

Notice some general properties of T^h : $|T^h(f)| \leq |f|$ and $|T^h(f)| \leq r(h)$, for all $f \in \mathbb{R}^n$. The pure projection also satisfies $T^h(f) = f$ for $|f| \leq r(h)$.

Notice that for both the case of the standard projection (a) and the case of the particular smoothed projection (b), the taming can be written multiplicatively, $f^h(t, y, z) = \chi^h(t, y, z) f(t, y, z)$. Indeed, we have

$$T^h(f) = \frac{1}{\max(1, |f| r(h)^{-1})}\, f \qquad \text{and} \qquad T^h(f) = \frac{1}{1 + |f| r(h)^{-1}}\, f$$

in cases (a) and (b), respectively.

Case (a), the standard projection, can therefore be viewed as a generalization of what we did for the verifications that the multiplicatively tamed drivers fit in our framework, the generalization being two-fold : first, we consider a factor $\chi^h(t, y, z)$ instead of just $\chi^h(y)$, second, we have to deal with $\max(1, x)$ instead of $1 + x$.

For case (b), this second generalization is non-existent. And if we consider a driver depending only on y, as we do in all our examples, we see that the outer taming (b) was already treated as the multiplicative taming (a). So let us ignore this case and focus only on the standard projection, in this subsection.

From now on, $T = T^h$ is the standard projection on the ball of radius $r = r(h)$.

2.2.1 Verification of (TGr)

By very construction,

$$|f^h(t, y, z)| = \big|T^h\big(f^h(t, y, z)\big)\big| \leq r(h).$$

So we take $K_t^h = r(h)$, $K_y^h = 0$ and $K_z^h = 0$. We have $(K_t^h)^2 h$ bounded as $h \to 0$ iff $\beta \leq \frac{1}{2}$.

2.2.2 Verification of (TMonGr)

We write

$$\begin{aligned}
\big\langle y, f^h(t, y, z)\big\rangle &= \chi^h(t, y, z)\, \langle y, f(t, y, z)\rangle \\
&\leq \chi^h(t, y, z)\Big(\bar{M}_t + \bar{M}_y |y|^2 + \bar{M}_z |z|^2\Big) \\
&\leq \bar{M}_t + \chi^h(t, y, z) \bar{M}_y |y|^2 + \bar{M}_z |z|^2 \\
&\leq \bar{M}_t^h + \bar{M}_y^h |y|^2 + \bar{M}_z^h |z|^2
\end{aligned}$$

where $\bar{M}_t^h = \bar{M}_t$, $\bar{M}_z^h = \bar{M}_z$ and $\bar{M}_y^h = \max(0, \bar{M}_y)$. These constants do not depend on h, so they work for **(TMonGr)** (they are bounded as $h \to 0$).

2.2.3 Verification of (TReg)

Here, rather than going into potentially brutal computations, it is easier to remember that, since T^h is a projection on a ball, it is 1-Lipchitz. At least, I believe it should be, for visual reasons. We can however prove easily that it is 2-Lipschitz.

<digression> For f and f' in $\mathbb{R}^n$, we have 3 cases to consider : they are both inside the ball of radius r, both are not, or only one of them is not. If they are both inside the ball, then $|T(f') - T(f)| = |f' - f|$, which is good. If both of them are not,

$$\begin{aligned}|T(f') - F(f)| &= \left|\frac{rf'}{|f'|} - \frac{rf}{|f|}\right| = \left|\frac{r|f|f' - r|f'|f}{|f'||f|}\right| = \left|\frac{r|f|(f' - f) + r(|f| - |f'|)f}{|f'||f|}\right| \\ &\leq \frac{r|f||f' - f|}{|f'||f|} + \frac{r\,||f'| - |f'||\,|f|}{|f'||f|} = \frac{r}{|f'|}|f' - f| + \frac{r}{|f'|}\,||f'| - |f'|| \\ &\leq \frac{r}{|f'|}|f' - f| + \frac{r}{|f'|}|f' - f| \leq 2|f' - f|,\end{aligned}$$

using the reverse triangle inequality and $r < |f'|$. If only one of them is not, by symmetry, we can assume $|f'| > r \geq |f|$. Then,

$$\begin{aligned}|T(f') - F(f)| &= \left|\frac{rf'}{|f'|} - f\right| = \left|\frac{rf'}{|f'|} - \frac{|f|f}{|f|}\right| = \left|\frac{r|f|f' - |f'||f|f}{|f'||f|}\right| = \left|\frac{(r - |f'|)|f|f' + |f'||f|(f' - f)}{|f'||f|}\right| \\ &\leq \frac{|(r - |f'|)|\,|f||f'|}{|f'||f|} + \frac{|f'||f|\,|f' - f|}{|f'||f|} = (|f'| - r) + |f' - f| \leq 2|f' - f|,\end{aligned}$$

since $|f| \leq r < |f'|$ implies that $0 < |f'| - r \leq |f'| - |f| \leq ||f'| - |f|| \leq |f' - f|$. </digression>

Using **(Reg)** we have

$$\begin{aligned}\left|f^h(t', y, z') - f^h(t, y, z)\right| &= \left|T^h\big(f(t', y, z')\big) - T^h\big(f(t, y, z)\big)\right| \\ &\leq 2\,|f(t', y, z') - f(t, y, z)| \\ &\leq 2L_t|t' - t|^{\frac{1}{2}} + 2L_z|z' - z|.\end{aligned}$$

So **(TReg)** is satisfied with $L_t^h = 2L_t$ and $L_z^h = 2L_z$.

2.2.4 Verification of (TRegY)

Given that the outer taming by projection can be viewed as a generalization of the multiplicative taming, one approach is to verify **(TRegY)** by similar estimations. As previously, we write

$$\begin{aligned}f^h(t, y', z) - f^h(t, y, z) &= \chi^h(t, y', z)f(t, y', z) - \chi^h(t, y, z)f(t, y, z) \\ &= \chi^h(t, y', z)\chi^h(t, y, z)\big(f(t, y', z) - f(t, y, z)\big) + \mathcal{R}^{(\mathrm{regY})}(t, y, y', z),\end{aligned}$$

where

$$\mathcal{R}^{(\mathrm{regY})}(t, y, y', z) := \chi^h(t, y', z)\big(1 - \chi^h(t, y, z)\big)f(t, y', z) - \big(1 - \chi^h(t, y', z)\big)\chi^h(t, y, z)f(t, y, z).$$

We first estimate the "good term", that will give the Lipschitz-in-Y regularity for fixed h, and then will estimate the remainder term. Using **(TRegY)** , we have

$$\begin{aligned}|\chi^h(t, y', z)\chi^h(t, y', z)\big(f(t, y', z) - f(t, y, z)\big)| &\leq \chi^h(t, y', z)\chi^h(t, y', z)L_y\big(1 + |y'|^{m-1} + |y|^{m-1}\big)|y' - y| \\ &\leq L_y\big(1 + \chi^h(t, y', z)|y'|^{m-1} + \chi^h(t, y', z)|y|^{m-1}\big)|y' - y|.\end{aligned}$$

To estimate the terms $\chi^h(t,y,z)|y|^{m-1}$ we need to introduce the assumption that "$|y|^m$ is dominant in f, uniformly in t and z" : there exist $k_y \geq 0$ and $R \geq 1$ such that, for all $y : |y| \geq R$, for t and z, $|f(t,y,z)| \geq k_y|y|^m$.

$$\chi^h(t,y,z)|y|^{m-1} = \frac{|y|^{m-1}}{\max(1,|f(t,y,z)|r(h)^{-1})} \leq \begin{cases} \frac{|y|^{m-1}}{k_y|y|^m}r(h) & \text{if } |y| \geq R \geq 1 \\ R^{m-1} & \text{if } |y| \leq R \end{cases} \leq R^{m-1} + \frac{1}{k_y}r(h).$$

So we take $L_y^h := L_y\big(1 + 2R^{m-1} + \frac{2}{k_y}r(h)\big)$. And $(L_y^h)^2h$ is bounded iff $\beta \leq \frac{1}{2}$.

It remains to estimate the remainder term. We note before starting that $\chi^h(t,y,z) \in [0,1]$ and that since, for $x \geq 0$, $\max(1,x) - 1 \leq x$, we have

$$1 - \chi^h(t,y,z) = \frac{\max(1,|f(t,y,z)|r(h)^{-1}) - 1}{\max(1,|f(t,y,z)|r(h)^{-1})} \leq \chi^h(t,y,z)|f(t,y,z)|r(h)^{-1} \leq |f(t,y,z)|r(h)^{-1}.$$

Then, we estimate

$$\begin{aligned} |\mathcal{R}^{(\mathrm{regY})}(t,y,y',z)| &= \big|\chi^h(t,y',z)\big(1-\chi^h(t,y,z)\big)f(t,y',z) - \big(1-\chi^h(t,y',z)\big)\chi^h(t,y,z)f(t,y,z)\big| \\ &\leq \chi^h(t,y',z)\big(1-\chi^h(t,y,z)\big)\,|f(t,y',z)| + \big(1-\chi^h(t,y',z)\big)\chi^h(t,y,z)\,|f(t,y,z)| \\ &\leq \Big\{1 \times |f(t,y,z)| \times |f(t,y',z)| + |f(t,y',z)| \times 1 \times |f(t,y,z)|\Big\}r(h)^{-1}. \end{aligned}$$

Using **(Gr)** we therefore have

$$\begin{aligned} |\mathcal{R}^{(\mathrm{regY})}(t,y,y',z)| &\leq 2|f(t,y,z)||f(t,y',z)|r(h)^{-1} \\ &\leq C\Big\{\big(1+|y'|^m+|z|\big)\big(1+|y|^m+|z|\big)\Big\}r(h)^{-1} \\ &\leq C\Big\{1+|y|^{2m}+|y'|^{2m}+|z|^2\Big\}h^\beta. \end{aligned}$$

To obtain the last estimate we have used several times the inequality $ab \leq a^2 + b^2$. Therefore, **(TRegY)** is verified.

The above estimation followed the template of multiplicative tamings but is suboptimal here. Instead, let us use the fact that the projection T^h leaves the ball of radius $r(h)$ invariant to obtain a more appropriate estimate.

Using the fact that $T(f) = f$ if $|f| \leq r$, we write

$$\begin{aligned} |f^h(t,y',z) - f^h(t,y,z)| &= |f(t,y',z) - f(t,y,z)|\ 1_{\{|f(t,y',z)| \leq r(h) \text{ and } |f(t,y,z)| \leq r(h)\}} \\ &\quad + |f^h(t,y',z) - f^h(t,y,z)|\ 1_{\{|f(t,y',z)| > r(h) \text{ or } |f(t,y,z)| > r(h)\}}. \end{aligned}$$

The first term is the "good term". Using **(RegY)** we see that

$$\begin{aligned} &|f(t,y',z) - f(t,y,z)|\ 1_{\{|f(t,y',z)| \leq r(h) \text{ and } |f(t,y,z)| \leq r(h)\}} \\ &\qquad \leq L_y\big(1+|y'|^{m-1}+|y|^{m-1}\big)|y'-y|\ 1_{\{|f(t,y',z)| \leq r(h) \text{ and } |f(t,y,z)| \leq r(h)\}} \\ &\qquad \leq L_y\big(1+|y'|^{m-1}1_{\{|f(t,y',z)| \leq r(h)\}}+|y|^{m-1}1_{\{|f(t,y,z)| \leq r(h)\}}\big)|y'-y|. \end{aligned}$$

It now remains to argue, using the fact that "$|y|^m$ is dominant in f, uniformly in t and z", that when $f(t,y,z)$ is not too large, y cannot be too large either (since the assumption implies that for large y, $f(t,y,z)$ must be large). To be precise, we know that there exist $k_y \geq 0$ and $R \geq 1$ such that, for all y such that $|y| \geq R$, for all t and z, $|f(t,y,z)| \geq k_y|y|^m$. Consider now $R' = \max\Big(R, \big(\frac{1}{k_y}r(h)\big)^{\frac{1}{m}}\Big) \geq R$. For $|y| > R'$, we have

$$r(h) < k_y|y|^m \leq |f(t,y,z)|.$$

Hence,

$$\begin{aligned}
1_{\{|f(t,y,z)|\le r(h)\}} &= 1_{\{|f(t,y,z)|\le r(h)\}}\ 1_{\{|y|\le R'\}} + 1_{\{|f(t,y,z)|\le r(h)\}}\ 1_{\{|y|> R'\}} \\
&= 1_{\{|f(t,y,z)|\le r(h)\}}\ 1_{\{|y|\le R'\}} + 0 \\
&\le 1_{\{|y|\le R'\}}.
\end{aligned}$$

Consequently, since $R' \ge R \ge 1$,

$$\begin{aligned}
L_y\Big(1 + |y'|^{m-1} 1_{\{|f(t,y',z)|\le r(h)\}} + |y|^{m-1} 1_{\{|f(t,y,z)|\le r(h)\}}\Big)& \\
&\le L_y\Big(1 + |y'|^{m-1} 1_{\{|y'|\le R'\}} + |y|^{m-1} 1_{\{|y|\le R'\}}\Big) \\
&\le L_y\Big(1 + (R')^{m-1} + (R')^{m-1}\Big) \\
&\le L_y\Big(1 + 2(R')^{m}\Big) \\
&\le L_y\Big(1 + \frac{2}{k_y} r(h)\Big) =: L_y^h.
\end{aligned}$$

And $(L_y^h)^2 h$ is bounded iff $\beta \le \frac{1}{2}$.

The second term is the remainder,

$$\mathcal{R}^{(\mathrm{regY})}(t,y',y,z) = \Big(T^h\big(f(t,y',z)\big) - T^h\big(f(t,y,z)\big)\Big)\ 1_{\{|f(t,y',z)|>r(h) \text{ or } |f(t,y,z)|>r(h)\}}$$

Using the fact that T^h is 2-Lipschitz, then **(RegY)** , we have

$$\begin{aligned}
|\mathcal{R}^{(\mathrm{regY})}(t,y',y,z)| &\le 2|f(t,y',z) - f(t,y,z)|\ 1_{\{|f(t,y',z)|>r(h) \text{ or } |f(t,y,z)|>r(h)\}} \\
&\le 2L_y\big(1 + |y'|^{m-1} + |y|^{m-1}\big)|y'-y|\ 1_{\{|f(t,y',z)|>r(h) \text{ or } |f(t,y,z)|>r(h)\}} \\
&\le C\big(1 + |y'|^{2m} + |y|^{2m}\big)\ 1_{\{|f(t,y',z)|>r(h) \text{ or } |f(t,y,z)|>r(h)\}}.
\end{aligned}$$

So $\mathcal{R}^{(\mathrm{regY})}$ satifies **(TCvg)** with the criterion 2.

2.2.5 Verification of (TMon)

Here again, given that the outer taming by projection can be viewed as a generalization of the multiplicative taming, one approach is to verify **(TMon)** by similar estimations.

We first write

$$\begin{aligned}
f^h(t,y,z) &= f(t,y,z) + (\chi^h(t,y,z) - 1)f(t,y,z) \\
&= f(t,y,z) - R^h(t,y,z).
\end{aligned}$$

Then, using **(Mon)** for f,

$$\begin{aligned}
\big\langle y'-y, f^h(t,y',z) - f^h(t,y,z)\big\rangle &= \langle y'-y, f(t,y',z) - f(t,y,z)\rangle \\
&\quad - \big\langle y'-y, (\chi^h(t,y',z)-1)f(t,y',z) - (\chi^h(t,y,z)-1)f(t,y,z)\big\rangle \\
&\le M_y\,|y'-y|^2 + \mathcal{R}^{(\mathrm{mon})}(t,y',y,z),
\end{aligned}$$

where

$$\mathcal{R}^{(\mathrm{mon})}(t,y',y,z) = -\big\langle y'-y, (\chi^h(t,y',z)-1)f(t,y',z) - (\chi^h(t,y,z)-1)f(t,y,z)\big\rangle.$$

We now estimate $\mathcal{R}^{(\mathrm{mon})}$ similarly to what we did to verify **(TRegY)** .

$$
\begin{aligned}
|\mathcal{R}^{(\mathrm{mon})}(t,y',y,z)| &\leq |y'-y| \Big[(1-\chi^h(t,y',z))|f(t,y',z)| + (1-\chi^h(t,y,z))|f(t,y,z)| \Big] \\
&\leq |y'-y| \Big[|f(t,y',z)| r(h)^{-1} |f(t,y',z)| + |f(t,y,z)| r(h)^{-1} |f(t,y,z)| \Big] \\
&\leq (|y'|+|y|) \Big[|f(t,y',z)|^2 + |f(t,y,z)|^2 \Big] r(h)^{-1} \\
&\leq C\big(|y'|+|y|\big) \Big[\big(1+|y'|^m+|z|\big)^2 + \big(1+|y|^m+|z|\big)^2 \Big] h^\beta \\
&\leq C\big(|y'|+|y|\big) \Big[1+|y'|^{2m}+|y|^{2m}+|z|^2 \Big] h^\beta \\
&\leq C \Big[1+|y'|^{4m}+|y|^{4m}+|z|^4 \Big] h^\beta .
\end{aligned}
$$

Again, the last inequality resulted from using several Young inequalities and $|y|^p \leq 1+|y|^q$ for $q \geq p$. Remark in particular that z appears with a power 4, which is not so good.

Therefore, **(TMon)** is verified.

This paragraph follows the 2nd and 3rd paragraph of the next subsubsection, using an estimation of $R^h(t,y,z)$ more appropriate to the projection. So it probably should be read afterwards. As we have written above already,

$$
f^h(t,y,z) = f(t,y,z) - R^h(t,y,z).
$$

Then, using **(Mon)** for f,

$$
\begin{aligned}
\big\langle y'-y, f^h(t,y',z) - f^h(t,y,z) \big\rangle = \langle y'-y, f(t,y',z) - f(t,y,z) \rangle& \\
- \big\langle y'-y, R^h(t,y',z) - R^h(t,y,z) \big\rangle& \\
\leq M_y \, |y'-y|^2 + \mathcal{R}^{(\mathrm{mon})}(t,y',y,z)&,
\end{aligned}
$$

where

$$
\mathcal{R}^{(\mathrm{mon})}(t,y',y,z) = - \big\langle y'-y, R^h(t,y',z) - R^h(t,y,z) \big\rangle .
$$

With a force as brute as before, but using now the better estimate for R^h, we have

$$
\begin{aligned}
|\mathcal{R}^{(\mathrm{mon})}(t,y',y,z)| &\leq |y'-y| \Big[|R^h(t,y',z)| + |R^h(t,y,z)| \Big] \\
&\leq C(|y'|+|y|) \Big[\big(1+|y'|^m+|z|\big) 1_{\{|f(t,y',z)|>r(h)\}} + \big(1+|y|^m+|z|\big) 1_{\{|f(t,y,z)|>r(h)\}} \Big] \\
&\leq C\big(|y'|+|y|\big) \Big[1+|y'|^m+|y|^m+|z| \Big] 1_{\{|f(t,y',z)|>r(h) \text{ or } |f(t,y,z)|>r(h)\}} \\
&\leq C \Big[1+|y'|^{2m}+|y|^{2m}+|z|^2 \Big] 1_{\{|f(t,y',z)|>r(h) \text{ or } |f(t,y,z)|>r(h)\}} .
\end{aligned}
$$

2.2.6 Verification of (TCvg)

Here again, given that the outer taming by projection can be viewed as a generalization of the multiplicative taming, one approach is to verify **(TMon)** by similar estimations.

$$
R^h(t,y,z) = f(t,y,z) - f^h(t,y,z) = f(t,y,z)(1-\chi^h(t,y,z)).
$$

So, using as before $1-\chi^h(t,y,z) \le |f(t,y,z)|r(h)^{-1}$,

$$\begin{aligned}\left|R^h(t,y,z)\right| &\le |f(t,y,z)|^2 r(h)^{-1}\\ &\le C(1+|y|^m+|z|)^2 h^\alpha\\ &\le C(1+|y|^{2m}+|z|^2)h^\alpha.\end{aligned}$$

Therefore f^h satisfies **(TCvg)** with the criterion 1.

Notice however that the above approach is not optimal ! Indeed, we know that the difference $R^h(t,y,z) = f(t,y,z) - f^h(t,y,z)$ is null when $f(t,y,z)$ is not too large. We should aim at estimating $R^h(t,y,z)$ with criterion 2 of **(TCvg)** . This has the major advantage that one can then use the Markov inequality and essentially get the convergence rate we want. Notice also that the first-approach estimation of $\mathcal{R}^{(\mathrm{mon})}$, following what was done for multiplicative tamings, in fact contains an estimation of $R^h(t,y,z)$. So we would improve the estimate there as well, and have a better chance at proving that the scheme has the convergence rate we want.

Using the fact that $T^h(f) = f$ for $|f| \le r(h)$, we see that

$$R^h(t,y,z) = \Big[f(t,y,z) - T^h\big(f(t,y,z)\big)\Big]1_{\{|f(t,y,z)|>r(h)\}}.$$

Here we can save a factor 2 in the constants by using the fact $|f - T^h(f)| \le |f|$ and so with **(Gr)** we have

$$\begin{aligned}\left|R^h(t,y,z)\right| &\le \left|f(t,y,z) - T^h\big(f(t,y,z)\big)\right|\ 1_{\{|f(t,y,z)|>r(h)\}}\\ &\le \left|f(t,y,z)\right|\ 1_{\{|f(t,y,z)|>r(h)\}}\\ &\le C\big(1+|y|^m+|z|\big)\ 1_{\{|f(t,y,z)|>r(h)\}}.\end{aligned}$$

Thus f^h satisfies **(TCvg)** with the criterion 2. As said above, this estimate can be used to better estimate $\mathcal{R}^{(\mathrm{mon})}$.

2.3 Verifications for the inner tamings

Consider a radius $r(h) = r_0 h^{-\gamma}$. The inner taming is given by

$$f^h(t,y,z) = f\big(t, T^h(y), z\big),$$

where T^h is the projection on the ball of $\mathbb{R}^n$ of center 0 and radius $r(h)$.

Recall the basic properties of T : $|T(y)| \le r$ and $|T(y)| \le |y|$ for all y, and $T(y) = y$ if $|y| \le r$.

2.3.1 Verification of (TGr)

Using **(Gr)** ,

$$|f^h(t,y,z)| \le K_t + K_y |T^h(y)|^m + K_z|z| \le K_t + K_y r(h)^{m-1}|y| + K_z|z|.$$

So we set $K_t^h = K_t$, $K_y^h = K_y f(h)^{m-1}$ and $K_z^h = K_z$. We have $(K_y^h)^2 h$ bounded iff $\gamma \le \frac{1}{2(m-1)}$.

2.3.2 Verification of (TMonGr)

We write

$$\begin{aligned}
\big\langle y, f^h(t,y,z)\big\rangle &= \big\langle y, f(t,T^h(y),z)\big\rangle \\
&= \big\langle y, f(t,T^h(y),z) - f(t,0,z)\big\rangle + \langle y, f(t,0,z)\rangle \\
&\le \big\langle y-0, f(t,T^h(y),z) - f(t,0,z)\big\rangle + \alpha|y|^2 + \frac{K_t^2}{2\alpha} + \frac{K_z^2}{2\alpha}|z|^2,
\end{aligned}$$

by the standard manipulations using **(Gr)** . We now want handle the main term. If $|y| \le r(h)$, then by **(Mon)**

$$\big\langle y-0, f(t,T^h(y),z) - f(t,0,z)\big\rangle = \langle y-0, f(t,y,z) - f(t,0,z)\rangle \le M_y|y|^2 \le \max(0,M_y)|y|^2.$$

If $|y| > r(h)$, then we note that $T^h(y) = r(h)\frac{y}{|y|}$, or equivalently $y = \frac{|y|}{r(h)}T^h(y)$. Hence, from **(Mon)** we have

$$\begin{aligned}
\big\langle y-0, f(t,T^h(y),z) - f(t,0,z)\big\rangle &= \frac{|y|}{r(h)}\big\langle T^h(y)-0, f(t,T^h(y),z) - f(t,0,z)\big\rangle \\
&\le \frac{|y|}{r(h)} M_y |T^h(y)|^2 \\
&= \frac{|y|}{r(h)} M_y r(h)^2 = M_y r(h)|y| \\
&\le \max(0,M_y)|y|^2.
\end{aligned}$$

since $r < |y|$. So we have the same upper bound in both cases, and we can conclude that

$$\big\langle y, f^h(t,y,z)\big\rangle \le (\max(0,M_y)+\alpha)|y|^2 + \frac{K_t^2}{2\alpha} + \frac{K_z^2}{2\alpha}|z|^2.$$

Therefore, taking $\bar{M}_t^h = \frac{K_t^2}{2\alpha}$, $\bar{M}_y^h = \max(0,M_y)+\alpha$ and $\bar{M}_z = \frac{K_z^2}{2\alpha}$ suits.

2.3.3 Verification of (TReg)

We see immediately that, since f^y only alters the argument y, **(Reg)** is unchanged :

$$|f^h(t',y,z') - f^h(t,y,z)| = |f(t',T^h(y),z') - f(t,T^h(y),z)| \le L_t|t'-t|^{\frac{1}{2}} + L_z|z'-z|.$$

2.3.4 Verification of (TRegY)

Using **(RegY)** and the 2-Lipschitzness of T^h, we have

$$\begin{aligned}
|f^h(t,y',z) - f^h(t,y,z)| &= |f(t,T^h(y'),z) - f(t,T^h(y),z)| \\
&\le L_y\big(1 + |T^h(y')|^{m-1} + |T^h(y)|^{m-1}\big)|T^h(y') - T^h(y)| \\
&\le 2L_y\big(1 + r(h)^{m-1} + r(h)^{m-1}\big)|y'-y| \\
&= 2L_y\big(1 + 2r(h)^{m-1}\big)|y'-y|,
\end{aligned}$$

so we set $L_y^h = 2L_y\big(1+2r(h)^{m-1}\big)$. As usual, $(L_y^h)^2 h$ is bounded iff $\gamma \le \frac{1}{2(m-1)}$.

2.3.5 Verification of (TMon)

Sadly, it does not seem to be true that f^h satisfies **(Mon)** i.e. has $\mathcal{R}^{(\mathrm{mon})} = 0$. We can however do the standard estimation via R^h.

$$f^h(t,y,z) = f(t,y,z) - R^h(t,y,z).$$

Then, using **(Mon)** for f,

$$\begin{aligned}
\big\langle y'-y, f^h(t,y',z) - f^h(t,y,z)\big\rangle &= \langle y'-y, f(t,y',z) - f(t,y,z)\rangle \\
&\quad - \big\langle y'-y, R^h(t,y',z) - R^h(t,y,z)\big\rangle \\
&\le M_y\,|y'-y|^2 + \mathcal{R}^{(\mathrm{mon})}(t,y',y,z),
\end{aligned}$$

where

$$\mathcal{R}^{(\mathrm{mon})}(t,y',y,z) = -\big\langle y'-y, R^h(t,y',z) - R^h(t,y,z)\big\rangle.$$

Using the estimate for R^h found in the next subsubsection,

$$\begin{aligned}
|\mathcal{R}^{(\mathrm{mon})}(t,y',y,z)| &\le |y'-y||R^h(t,y',z) - R^h(t,y,z)| \\
&\le \big(|y'|+|y|\big)\big(|R^h(t,y',z)| + |R^h(t,y,z)|\big) \\
&\le C\big(|y'|+|y|\big)\Big(\big(1+|y'|^m\big)\ 1_{\{|y'|>r(h)\}} + \big(1+|y|^m\big)\ 1_{\{|y|>r(h)\}}\Big) \\
&\le C\big(|y'|+|y|\big)\Big(1+|y'|^m+|y|^m\Big)\ 1_{\{|y'|>r(h)\text{ or }|y|>r(h)\}} \\
&\le C\Big(1+|y'|^{2m}+|y|^{2m}\Big)\ 1_{\{|y'|>r(h)\text{ or }|y|>r(h)\}}.
\end{aligned}$$

2.3.6 Verification of (TCvg)

Using **(RegY)** and the fact that $|y - T^h(y)| \le |y|$, we have

$$\begin{aligned}
|R^h(t,y,z)| &= |f(t,y,z) - f^h(t,y,z)| \\
&= |f(t,y,z) - f(t,T^h(y),z)|\ 1_{\{|y|>r(h)\}} \\
&\le L_y\big(1+|y|^{m-1} + |T^h(y)|^{m-1}\big)|y - T^h(y)|\ 1_{\{|y|>r(h)\}} \\
&\le L_y\big(1+|y|^{m-1}+|y|^{m-1}\big)|y|\ 1_{\{|y|>r(h)\}} \\
&\le C\big(1+|y|^m\big)\ 1_{\{|y|>r(h)\}}.
\end{aligned}$$

2.4 Summary of the verifications and final convergence rates

2.4.1 The multiplicative tamings

It is seen relatively easily that **(TGr)** , **(TMonGr)** and **(TReg)** are satisfied. It takes some effort for **(TGr)** because there are 4 cases to treat, but the taming is designed to control the size of the driver, so **(TGr)** has to hold. The assumptions that are potentially delicate to verify are **(TRegY)** and **(TMon)** . The idea here is to identify a good term and a remainder. For the good term in **(TRegY)** , when the taming is based on the output $f(y)$ (rather than $|y|^m$ or $|y|^{m-1}$), one very naturally has to assume that "$|y|^m$ is dominant in f". One can very probably construct counter-examples to show that this assumption is required, and not just convenient. For **(TMon)** , the remainder term is very related to the remainder R^h.

The conditions on the constants (K_y^h, L_y^h, etc) require $\alpha \leq \frac{1}{2}$.

The estimates on the remainders $\mathcal{R}^{(\mathrm{regY})}$ and $\mathcal{R}^{(\mathrm{mon})}$ are as follow.

$$|\mathcal{R}^{(\mathrm{regY})}(t, y', y, z)| \leq C\Big(1 + |y'|^{2m} + |y|^{2m} + |z|^2\Big)h^{\alpha}$$

and

$$|\mathcal{R}^{(\mathrm{mon})}(t, y', y, z)| \leq C\Big(1 + |y'|^{4m} + |y|^{4m} + |z|^2\Big)h^{\alpha}.$$

The estimate on the convergence remainder is the following.

$$|R^h(t, y, z)| \leq C\Big(1 + |y|^{2m} + |z|^2\Big)h^{\alpha}.$$

According to lemma **??** for **(TCvg)** case 1, since we have found $p = 2$ and $\alpha \leq \frac{1}{2}$, we have

$$\mu = \alpha \leq \frac{1}{2}.$$

This means the global rate of the scheme is $r = \frac{1}{2}\min(1, \mu) = \frac{1}{4}$. This means the taming vanishes slower than the effect of the time-discretization, and the scheme overall has order 1/4 only. Having no Z-dependence does not help here, the limiting factor really is the fact that $\alpha = \alpha$ in **(TCvg)** .1 for $\mathcal{R}^{(\mathrm{mon})}$ (related to μ_2). This slowing-down might be due to the fact that, while the multiplicative taming with $1 + \ldots$ in the denominator of χ^h helps controlling the driver outside the ball of radius $r(h)$, it also induces an error inside the ball, and that slows down the convergence of the scheme.

2.4.2 The outer taming

Assumptions **(TGr)** , **(TMonGr)** and **(TReg)** are almost immediate to verify. The assumptions that are potentially delicate to verify are **(TRegY)** and **(TMon)** , and the good way is not to view the projection as a multiplicative taming but really use the fact that projection restricted to the ball of radius $r(h) = r_0 h^{-\beta}$ is the identity. The idea is to distinguish the cases $f(t, y, z)$ is in the ball and $f(t, y, z)$ is out of the ball. It is again naturally (and most probably necessarily) that the assumption "$|y|^m$ is dominant in f" is used, as the taming is done according to the size of the output $f(t, y, z)$.

The conditions on the constants (K_y^h, L_y^h, etc) require $\beta \leq \frac{1}{2}$.

The estimates on the remainders $\mathcal{R}^{(\mathrm{regY})}$ and $\mathcal{R}^{(\mathrm{mon})}$ are as follow.

$$|\mathcal{R}^{(\mathrm{regY})}(t,y',y,z)| \leq C\Big(1+|y'|^{2m}+|y|^{2m}\Big)\ 1_{\{|f(t,y',z)|>r(h) \text{ or } |f(t,y,z)|>r(h)\}}$$

and

$$|\mathcal{R}^{(\mathrm{mon})}(t,y',y,z)| \leq C\Big(1+|y'|^{2m}+|y|^{2m}+|z|^2\Big)\ 1_{\{|f(t,y',z)|>r(h) \text{ or } |f(t,y,z)|>r(h)\}}.$$

The estimate on the convergence remainder is the following.

$$|R^h(t,y,z)| \leq C\Big(1+|y|^m+|z|\Big)\ 1_{\{|f(t,y,z)|>r(h)\}}.$$

According to lemma **??** for **(TCvg)** case 2, since we have found, for $\mathcal{R}^{(\mathrm{regY})}$, $p_{\mathrm{regY}} = 0$ (no z in the remainder), and for $\mathcal{R}^{(\mathrm{mon})}$, $p_{\mathrm{mon}} = 2$, with $\beta = \frac{1}{2}$, we have

$$\mu = \frac{\beta l}{2} = \frac{l}{4}.$$

This means the global rate of the scheme is $r = \frac{1}{2}\min(1,\mu) = \frac{1}{2}$, since l can be chosen arbitrarily large. The effect of taming therefore vanishes arbitrarily fast, and surely faster than the error induced by the time-discretization.

2.4.3 The inner taming

Those verifications are globally easier than for the outer taming. Assumptions **(TGr)** , **(TMonGr)** and **(TReg)** are almost immediate to verify, **(TMonGr)** being the only one requiring real arguments. **(TRegY)** is easy to verify and is actually satisfied with $\mathcal{R}^{(\mathrm{regY})} = 0$.

The conditions on the constants (K_y^h, L_y^h, etc) require $\gamma \leq \frac{1}{2(m-1)}$.

The estimates on the remainders $\mathcal{R}^{(\mathrm{regY})}$ and $\mathcal{R}^{(\mathrm{mon})}$ are as follow.

$$|\mathcal{R}^{(\mathrm{regY})}(t,y',y,z)| = 0$$

and

$$|\mathcal{R}^{(\mathrm{mon})}(t,y',y,z)| \leq C\Big(1+|y'|^{2m}+|y|^{2m}\Big)\ 1_{\{|y'|>r(h) \text{ or } |y|>r(h)\}}.$$

The estimate on the convergence remainder is the following.

$$|R^h(t,y,z| \leq C\Big(1+|y|^m\Big)\ 1_{\{|y|>r(h)\}}.$$

According to lemma **??** for **(TCvg)** case 2, since we have found, $\mathcal{R}^{(\mathrm{regY})} = 0$, so we can take $\mu_1 = \mu_2 = +\infty$, and for $\mathcal{R}^{(\mathrm{mon})}$, $p_{\mathrm{mon}} = 0$ (no z in the remainder), with $\gamma = \frac{1}{2(m-1)}$, we have

$$\mu = \frac{\gamma l}{2} = \frac{l}{4(m-1)}.$$

But l can be chosen as big as wanted. This means the global rate of the scheme is $r = \frac{1}{2}\min(1,\mu) = \frac{1}{2}$, as usual. With or without a z dependence, the effect of taming therefore vanishes arbitrarily fast, and surely faster than the error induced by the time-discretization.