Stability and analytic expansions of local solutions of systems of quadratic BSDEs with applications to a price impact model

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Outline

Price impact model

Input: dividends, market makers' preferences, demand

1. Ψ: stocks' dividends paid at maturity T
2. $U(x) = -\frac{1}{a}e^{-ax}$, $x \in \mathbf{R}$: representative utility; $a > 0$ is aggregate risk-aversion
3. $\gamma = (\gamma_t)$: demand process (number of stocks)

Output: stocks' prices $S = S(\gamma, a) = (S_t)$ such that

$$\gamma = \arg\max_{\zeta} \mathbb{E}[U(\int_0^T \zeta dS)] = \arg\min_{\zeta} \mathbb{E}[\exp(-a \int_0^T \zeta dS)]$$

and $S_t = \mathbb{E}_t^{\mathbb{Q}}[\Psi]$, $t \in [0, T]$, with $\mathbb{Q} = \mathbb{Q}(\gamma, S)$ given by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \text{const}\, U'(\int_0^T \gamma dS) = \text{const}\exp(-a \int_0^T \gamma dS) \qquad (1)$$

References: Grossman and Miller (1988) (single period), Garleanu et al. (2009) (discrete time), German (2011) (simple strategy)

Example: Bachelier model - Simple strategies

Assume that $\Psi = \sigma B_T$ (B a $\mathbb{P}$-BM) and that $\gamma = q$ constant. We can rewrite our equilibrium mechanism as

$$S_t = \frac{\mathbb{E}_t[\Psi \exp(-aq\Psi)]}{\mathbb{E}_t[\exp(-aq\Psi)]}$$

In this case,

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E}(-aq\sigma B)_T$$

By Girsanov's theorem $B_t + aq\sigma t$ is a $\mathbb{Q}$-BM. Therefore

$$\begin{aligned} S_t &= \mathbb{E}_t^{\mathbb{Q}}[\Psi] = \mathbb{E}_t^{\mathbb{Q}}[\sigma B_T] = \sigma(B_t + aq\sigma t) - aq\sigma^2 T \\ &= S_t(0) - aq\sigma^2(T - t) \end{aligned} \tag{2}$$

In general for **simple demands, if Ψ has exponential moments, prices can be found by a backward recursion process!**

Brownian framework

Assumption

The filtration is generated by a Brownian motion $B = (B_t)$:

$$\mathcal{F}_t = \mathcal{F}_t^B, \quad t \in [0, T]$$

Stocks' prices evolve as

$$dS = \sigma \lambda dt + \sigma dB, \quad S_T = \Psi, \tag{3}$$

where

$\lambda = (\lambda_t)$: the market price of risk;
$\sigma = (\sigma_t)$: the volatility

Denote also

$$R_t = -\frac{1}{a} \log \mathbb{E}_t \left[\exp(-a \int_t^T \gamma dS) \right], \quad t \in [0, T],$$

the certainty equivalence value (CEV) of remaining gain

Multi-dimensional quadratic BSDE

Theorem

For dividends Ψ *and demand* γ*, the following items are equivalent:*

1. $S = S(\Psi, \gamma)$ *is a price process,* σ *is the volatility,* λ *is the market price of risk, and* R *is the CEV process*
2. (R, S, η, θ) *with* $\eta = \lambda - a\sigma\gamma$ *and* $\theta = a\sigma$ *solves the BSDE:*

$$aR_t = \frac{1}{2}\int_t^T (|\theta\gamma|^2 - |\eta|^2)ds - \int_t^T \eta dB, \tag{4}$$

$$aS_t = a\Psi - \int_t^T \theta(\eta + \theta\gamma)ds - \int_t^T \theta dB, \tag{5}$$

and the stochastic exponential

$$Z \triangleq \mathcal{E}(-\int \lambda dB) = \mathcal{E}(-\int (\eta + \theta\gamma)dB)$$

and the products $Z(\int \gamma dS)$ *and* ZS *are martingales*

BMO norms

- For a continuous martingale M with $M_0 = 0$,

$$\|M\|_{\mathrm{BMO}} \triangleq \sup_{\tau} \| \{\mathbb{E}_\tau[|M_T - M_\tau|^2]\}^{1/2} \|_\infty,$$

  where the supremum is taken with respect to all stopping times τ

- For an integrable random variable ξ set

$$\|\xi\|_{\mathrm{BMO}} \triangleq \|(\mathbb{E}_t[\xi] - \mathbb{E}[\xi])_{t \in [0,T]}\|_{\mathrm{BMO}}$$

- For a predictable process $\zeta = (\zeta_t)$ set

$$\|\zeta\|_{\mathrm{BMO}} \triangleq \sup_{\tau} \| \left(\mathbb{E}_\tau[\int_\tau^T |\zeta_s|^2 ds] \right)^{1/2} \|_\infty,$$

  where the supremum is taken with respect to all stopping times τ. By Ito's isometry,

$$\|\zeta\|_{\mathrm{BMO}} = \| \int \zeta dB \|_{\mathrm{BMO}}$$

Existence and uniqueness results

Theorem

There is a constant $c > 0$ *such that if*

$$a\|\gamma\|_\infty \|\Psi\|_{\mathrm{BMO}} \leq c, \tag{6}$$

then the prices $S = S(\Psi, \gamma)$ *exist and are unique. In this case*

$$\begin{aligned} \|\lambda\|_{\mathrm{BMO}} &\leq 4a\|\gamma\|_\infty \|\Psi\|_{\mathrm{BMO}} \\ \|\sigma\|_{\mathrm{BMO}} &\leq 2\|\Psi\|_{\mathrm{BMO}} \end{aligned} \tag{7}$$

Proposition

There are bounded γ *and* Ψ *such that*

$$a\|\gamma\|_\infty \|\Psi\|_\infty \leq 1$$

and such that the prices $S = S(\Psi, \gamma)$ *either do not exist or are not unique*

Questions

1. Suppose that $(\gamma_n)_{n\geq 1}$ simple and γ satisfy (6) and that

$$\gamma_n \to \gamma$$

Prices $S(\gamma_n)$ **can be found by backward recursion.** Do we have convergence of prices?

$$S(\gamma_n) \to S(\gamma) \tag{8}$$

2. By (7) if $a \to 0$, then $\lambda(a) \to 0$ and

$$S_t(a) \to S_t(0) = \mathbb{E}_t[\Psi] = \text{const} + (\sigma(0) \cdot B)_t$$

Can we write an asymptotic expansion of prices in terms of a?

$$S(a) = S(0) + \text{correction terms} \tag{9}$$

Local stability of systems of quadratic BSDEs - Setup

Consider the n-dimensional BSDEs:

$$Y_t = \Xi + \int_t^T f(s, \zeta_s)\, ds - \int_t^T \zeta\, dB, \quad t \in [0, T] \tag{10}$$

$$Y'_t = \Xi' + \int_t^T f'(s, \zeta'_s)\, ds - \int_t^T \zeta'\, dB, \quad t \in [0, T], \tag{11}$$

Assume that f, f' are quadratic,

$$|f(t,u) - f(t,v)| \leq \Theta(|u-v|)(|u| + |v|),$$
$$|f'(t,u) - f'(t,v)| \leq \Theta(|u-v|)(|u| + |v|),$$

$\Xi, \Xi' \in \mathrm{BMO}$ and there exists a nonnegative process $\delta = (\delta_t)$ such that

$$|f(t,z) - f'(t,z)| \leq \delta_t |z|^2$$

Auxiliary definitions: p–norms

- $\mathcal{S}_p(\mathbf{R}^n)$: Semimartingales $X = X_0 + M + A$, where M is a continuous martingale and A is a process of finite variation, with the norm

$$\|X\|_{\mathcal{S}_p} \triangleq |X_0| + \|\langle M\rangle_T^{1/2}\|_{\mathcal{L}_p} + \|\int_0^T |dA|\|_{\mathcal{L}_p}$$

- $\mathcal{H}_p(\mathbf{R}^{n\times d})$: ζ predictable such that $\zeta \cdot B \in \mathcal{S}_p(\mathbf{R}^n)$ for a d-dimensional Brownian motion B. It is a Banach space under the norm:

$$\|\zeta\|_{\mathcal{H}_p} \triangleq \|\zeta \cdot B\|_{\mathcal{S}_p} = \{\mathbb{E}\left[\left(\int_0^T |\zeta_s|^2 ds\right)^{p/2}\right]\}^{1/p}$$

Local stability of systems of quadratic BSDEs

Theorem

Assume that the BSDEs (10)– (11) *satisfy the previous conditions and let* (Y,ζ) *and* (Y',ζ') *be their respective solutions. For* $p>1$ *there are positive constants* $c=c(n,p)$ *and* $C=C(n,p)$ *(depending only on* n *and* p*) such that if*

$$\|\zeta\|_{\mathrm{BMO}}+\|\zeta'\|_{\mathrm{BMO}}\leq\frac{c}{\Theta}, \tag{12}$$

then

$$\|\zeta'-\zeta\|_{\mathcal{H}_p}\leq C\left(\|\Xi'-\Xi\|_{\mathcal{L}_p}+\|\sqrt{\delta}\zeta\|^2_{\mathcal{H}_{2p}}\right), \tag{13}$$

$$\|Y'-Y\|_{\mathcal{S}_p}\leq C\left(\|\Xi'-\Xi\|_{\mathcal{L}_p}+\|\sqrt{\delta}\zeta\|^2_{\mathcal{H}_{2p}}\right) \tag{14}$$

Stability of prices with respect to demands

Theorem

There is a constant $c = c(n,p) > 0$ such that if $(\gamma^m)_{m\geq 1}$ and γ are bounded demands with

$$a\|\gamma^m\|_\infty \|\Psi\|_{\mathrm{BMO}} \leq c, \quad m \geq 1, \tag{15}$$

and

$$\mathbb{E}\left[\int_0^T |\gamma^m_t - \gamma_t| dt\right] \to 0, \quad n \to \infty,$$

then $(\gamma^m)_{m\geq 1}$ and γ are viable demands and the corresponding stock prices $(S^m)_{m\geq 1}$ and S, volatilities $(\sigma^m)_{m\geq 1}$ and σ, and the market prices of risk $(\lambda^m)_{m\geq 1}$ and λ converge as

$$\|S^m - S\|_{\mathcal{S}_p} + \|\sigma^m - \sigma\|_{\mathcal{H}_p} + \|\lambda^m - \lambda\|_{\mathcal{H}_p} \to 0, \quad m \to \infty. \tag{16}$$

In particular, prices can be well approximated by the prices originated from simple demands

Parametrized family of BSDEs

Consider an n-dimensional BSDE

$$Y_t = a\Xi + \int_t^T f(s, \zeta_s)\, ds - \int_t^T \zeta\, dB, \quad t \in [0, T], \tag{17}$$

where the terminal condition depends on a parameter $a \in \mathbf{R}$. There is only one solution $(Y(a), \zeta(a))$ such that $\|\zeta(a)\|_{\mathrm{BMO}}$ is small and for this solution we have an estimate:

$$\|\zeta(a)\|_{\mathrm{BMO}} \leq 2|a|\|\Xi\|_{\mathrm{BMO}}.$$

In particular, $\zeta(a)$ converges to 0 in BMO as a approaches 0

Analytic expansion of systems of purely quadratic BSDEs

Theorem

If $f(u) = \widetilde{f}(u, u)$ *(*$\widetilde{f}$ *bilinear) then, the solution* $(Y(a), \zeta(a))$ *to* (17) *has a power series expansion*

$$Y(a) = \sum_{k=1}^{\infty} Y^{(k)} a^k \quad \textit{and} \quad \zeta(a) = \sum_{k=1}^{\infty} \zeta^{(k)} a^k$$

convergent for $|a|$ *small in* BMO

Analytic expansion of systems of purely quadratic BSDEs

The coefficients can be calculated recursively by

$$Y_t^{(1)} = \mathbb{E}_t[\Xi], \quad t \in [0,T],$$
$$\zeta^{(1)} \cdot B = Y_t^{(1)} - Y_0^{(1)}$$

and for $k \geq 2$

$$\zeta^{(k)} = \sum_{l+m=k} \widetilde{F}(\zeta^{(l)}, \zeta^{(m)}), \tag{18}$$

$$Y_t^{(k)} = \sum_{l+m=k} \mathbb{E}_t[\int_t^T \widetilde{f}(s, \zeta_s^{(l)}, \zeta_s^{(m)}) ds], \quad t \in [0,T], \tag{19}$$

where

$$(\widetilde{F}(\mu,\nu) \cdot B)_t = \mathbb{E}_t\left[\int_0^T \widetilde{f}(s,\mu_s,\nu_s)\, ds\right] - \mathbb{E}\left[\int_0^T \widetilde{f}(s,\mu_s,\nu_s)\, ds\right]$$

Analytic expansions of prices with respect to a

Theorem

There is a constant $c = c(n) > 0$ *such that if*

$$0 < a < \rho \triangleq \frac{c}{\|\gamma\|_\infty \|\Psi - \mathbb{E}[\Psi]\|_{\mathrm{BMO}}},$$

then γ *is a viable demand. The price* $S(\gamma; a)$ *is unique and admits the power series expansion in* BMO*:*

$$S(\gamma; a) = S(0) + \sum_{k=1}^{\infty} S^{(k)} a^k, \quad a < \rho$$

The market price of risk $\lambda(\gamma; a)$ *and the volatility* $\sigma(\gamma; a)$ *also have the power series expansions in* BMO

First order approximation

The leading price impact coefficient in the expansion for stock prices is given by

$$S_t^{(1)} = -\mathbb{E}_t \left[\int_t^T \sigma_s(0)^2 \gamma_s \, ds \right], \quad t \in [0, T]$$

This result had been obtained earlier in German (2011) for a simple demand; see (2)

Summary

- We study the price impact model of Grossman and Miller (1988); inverse to optimal investment
- Equivalent to multi-dimensional quadratic BSDE
- Stability of prices with respect to demands. Approximation with simple demands
- Analytic expansion of prices with respect to risk aversion coefficient

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Thank you!