An adverse selection approach to power tarification

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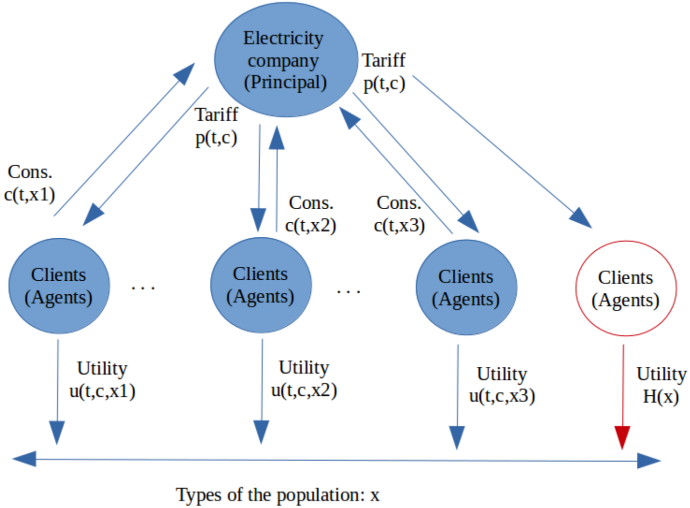
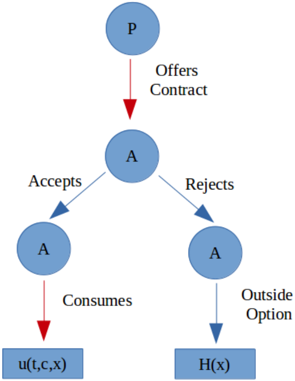
An electricity company wants to determine the optimal tariff $p(t, c)$ of the electrical consumption for its clients.



From the game theory point of view, the Company and the Client play a Non-zero sum Stackelberg game.



- $K : [0,T] \times \mathcal{C} \longrightarrow \mathbb{R}_+$ is the cost of production of electricity for the Principal. $(K(t,c))$
  - $t \mapsto K(t,c)$ is continuous $\forall c$.
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- x is the Agent's type, taking values in some set $X \subset \mathbb{R}$.
- $f : X \longrightarrow \mathbb{R}_+$ is the density of the Agent's type on X.
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- $u : [0, T] \times X \times \mathcal{C} \longrightarrow \mathbb{R}$ is the utility function of the Agents. $(u(t, x, c))$
  - u is jointly continuous.
  - $c \longmapsto u(t, x, c)$ is non-decreasing and concave for every (t, x).
  - $c \longmapsto \frac{\partial u}{\partial x}(t, x, c)$ is invertible.

Given $p \in \mathcal{P}$, the problem of the Agent of type $x \in X$ is

$$U_A(p,x) := \sup_c \int_0^T u(t,x,c(t)) - p(t,c(t)) \, \mathrm{d}t.$$

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Definition 1

Let φ be a map from $[0,T] \times \mathcal{C}$ to $\mathbb{R}$. The $u-$transform $\varphi^\star : [0,T] \times X \longrightarrow \mathbb{R} \cup \{+\infty\}$, is defined by

$$\varphi^\star(t,x) := \sup_{c \in \mathcal{C}} \{u(t,x,c) - \varphi(t,c)\}, \text{ for any } (t,x) \in [0,T] \times X.$$

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Similarly, if ψ is a map from $[0, T] \times X$ to $\mathbb{R}$, its $u-$transform $\psi^\star : [0, T] \times \mathcal{C} \longrightarrow \mathbb{R} \cup \{+\infty\}$ is defined by

$$\psi^\star(t, c) := \sup_{x \in X} \{u(t, x, c) - \psi(t, x)\}, \text{ for any } (t, c) \in [0, T] \times \mathcal{C}.$$

Definition 2

A map $\varphi : [0,T] \times \mathcal{C} \longrightarrow \mathbb{R} \cup \{+\infty\}$ is said to be $u-$convex if it is proper and if there exists some $\psi : [0,T] \times X \longrightarrow \mathbb{R}$ such that

$$\varphi(t,c) = \psi^{\star}(t,c), \text{ for any } (t,c) \in [0,T] \times \mathcal{C}.$$

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Proposition 1

A map $\varphi : [0,T] \times \mathcal{C} \longrightarrow \mathbb{R} \cup \{+\infty\}$ is $u-$convex if and only if

$$\varphi(t,c) = (\varphi^\star)^\star(t,c), \textit{ for any } (t,c) \in [0,T] \times \mathcal{C}.$$

$\Longrightarrow$ If p is u-convex it can be recovered from $p^\star$.

Definition 3

Let $\psi : [0,T] \times X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a $u-$convex function. For any $(t,x) \in [0,T] \times X$, the $u-$subdifferential of ψ at the point (t,x) is the set

$$\partial^\star \psi(t,x) := \{c \in \mathcal{C},\ \psi^\star(t,c) = u(t,x,c) - \psi(t,x)\}.$$

Definition 4

A tariff $p : [0,T] \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ is admissible if

- p is a $u-$convex function.
- $\partial^\star p^\star(t,x)$ is non-empty for every (t,x).
- $x \longmapsto p^\star(t,x) \in W^{1,m}(0,1)$, for a.e. t.

We denote by $\mathcal{P}$ the set of admissible tariffs.

For every $p \in \mathcal{P}$ the agent has a unique optimal response:

$$p^\star(t,x) = u(t,x,c^\star) - p(t,c^\star), \text{ for every } c^\star \in \partial^\star p^\star(t,x).$$

$$\Longrightarrow \frac{\partial u}{\partial x}(t,x,c^\star) = \frac{\partial p^\star}{\partial x}(t,x), \text{ for every } c^\star \in \partial^\star p^\star(t,x), \tag{1}$$

$$\Longrightarrow c^\star(t,x) = \left(\frac{\partial u}{\partial x}(t,x,\cdot)\right)^{(-1)} \left(\frac{\partial p^\star}{\partial x}(t,x)\right). \tag{2}$$

Obs: (1) $\Longrightarrow p^\star$ is increasing in x.

Proposition 2

For every $p \in \mathcal{P}$ *and for almost every* $x \in X$*, we have*

$$U_A(p,x) = \int_0^T p^\star(t,x)\mathrm{d}t,$$

and the optimal consumption of Agents is given by (2)*.*

The set of agents who accept the contract

$$X^\star(p^\star) := \left\{ x \in X,\ P^\star(x) := \int_0^T p^\star(t,x)\mathrm{d}t \geq H(x) \right\}.$$

The principal's problem is

$$U_P := \sup_{p\in\mathcal{P}} \int_0^T \left[\int_{X^\star(p^\star)} p(t, c^\star(t,x)) f(x)\mathrm{d}x - K\left(t, \int_{X^\star(p^\star)} c^\star(t,x) f(x)\mathrm{d}x \right) \right] \mathrm{d}t.$$

We solve actually $\widetilde{U}_P \geq U_P$ where we drop the u-convexity of p.

$$\widetilde{U}_P = \sup_{C^+} \int_0^T \left[\int_{X^\star(p^\star)} p(t, c^\star(t,x)) f(x) \mathrm{d}x - K\left(t, \int_{X^\star(p^\star)} c^\star(t,x) f(x) \mathrm{d}x \right) \right] \mathrm{d}t.$$

with C^+ is the space of maps $g : [0,T] \times X \longrightarrow \mathbb{R}$ such that

- $x \longmapsto g(t,x) \in W^{1,m}(0,1)$, for a.e. t.
- $x \longmapsto g(t,x)$ is non-decreasing.

We consider $X = [0, 1]$, and

$$u(t, x, c) = g(x)\phi(t)\frac{c^\gamma}{\gamma},$$

- $g : X \to \mathbb{R}_+$ continuous and non-decreasing.
- $\phi : [0, T] \longrightarrow \mathbb{R}_+^\star$ continuous.
- $\gamma \in (0, 1)$.

The response of the agent (2) can be written as

$$c^\star(t,x) = \left(\frac{\gamma}{\phi\,(t)\, g'(x)} \frac{\partial p^\star}{\partial x}\,(t,x) \right)^{\frac{1}{\gamma}}.$$

The principal's problem in terms of $p^\star$ is

$$\widetilde{U}_P = \sup_{p^\star \in C^+} \int_0^T \left[\int_{X^\star(p^\star)} \left(\frac{g(x)}{g'(x)} \frac{\partial p^\star}{\partial x}(t,x) - p^\star(t,x) \right) f(x)\mathrm{d}x \right. \\ \left. - K\left(t, \int_{X^\star(p^\star)} \left(\frac{\gamma}{\phi(t) g'(x)} \frac{\partial p^\star}{\partial x}(t,x) \right)^{\frac{1}{\gamma}} f(x)\mathrm{d}x \right) \right] \mathrm{d}t. \qquad (3)$$

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Recall that

$$X^\star(p^\star) = \{x \in [0,1] : P^\star(x) \geq H(x)\}\,.$$

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We study 3 cases for the reservation utility of the agents

1) H is constant.
2) H is strictly concave.
3) H is constant-linear.

1) H is constant. In this case $X^\star$ is an interval $[x_0, 1]$.

$$X^\star(p^\star) = \{x \in [0,1],\ P^\star(x) \geq H\}\,.$$

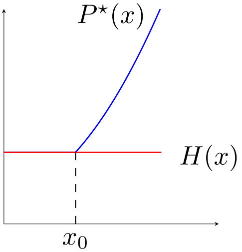

Figure : $X^\star(p^\star)$ for constant H.

We solve the equivalent formulation of (3)

$$\widetilde{U}_P = \sup_{x_0 \in [0,1]} \sup_{p^\star \in C^+(x_0)} \int_0^T \left[\int_{x_0}^1 \left(\frac{g(x)}{g'(x)} \frac{\partial p^\star}{\partial x}(t,x) - p^\star(t,x) \right) f(x)\mathrm{d}x \right. \\ \left. - K\left(t, \int_{x_0}^1 \left(\frac{\gamma}{\phi(t) g'(x)} \frac{\partial p^\star}{\partial x}(t,x) \right)^{\frac{1}{\gamma}} f(x)\mathrm{d}x \right) \right] \mathrm{d}t, \quad (4)$$

with

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i.e.

$$C^+(x_0) = \left\{ p^\star \in C^+ : \int_0^T p^\star(t, x_0)\, \mathrm{d}t = H \right\},$$

is a closed and convex set.

Which is equivalent, by integration by parts, to

$$\widetilde{U}_P = \sup_{x_0 \in [0,1]} \sup_{p^\star \in C^+(x_0)} \int_0^T \Bigg[\int_{x_0}^1 \frac{(g(x)f(x) + g'(x)F(x) - g'(x))}{g'(x)} \frac{\partial p^\star}{\partial x}(t,x)\mathrm{d}x - K\left(t, \int_{x_0}^1 \left(\frac{\gamma}{\phi(t)g'(x)} \frac{\partial p^\star}{\partial x}(t,x)\right)^{\frac{1}{\gamma}} f(x)\mathrm{d}x\right)\Bigg]\mathrm{d}t$$

$$+(F(x_0)-1)H.$$

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$$:= \Psi_{x_0}(p^\star)$$

We solve the following convex optimization problem

$$(P_{x_0}) \sup_{p^\star \in C^+(x_0)} \Psi_{x_0}(p^\star).$$

Sufficient and necessary optimality condition

$$\Psi'_{x_0}(p^\star; q) \le 0, \ \forall q \in T_{C^+(x_0)}(p^\star).$$

$$\Longrightarrow \ \frac{\partial p^\star}{\partial x}(t,x) = \left(\frac{\phi(t)^{\frac{1}{\gamma}} \left[g(x)f(x) + g'(x)F(x) - g'(x) \right]^+}{f(x)\frac{\partial K}{\partial c}\left(t, A(t,x_0)\right)} \right)^{\frac{\gamma}{1-\gamma}} \frac{g'(x)}{\gamma},$$

for a continuous map A.

The problem reduces to

$$\widetilde{U}_P = \sup_{x_0\in[0,1]} \int_0^T \left(\frac{1}{\gamma} \frac{\phi^{\frac{1}{1-\gamma}}(t)\ell(x_0)}{\left(\frac{\partial K}{\partial c}(t, A(t,x_0))\right)^{\frac{\gamma}{1-\gamma}}} - K\left(t, \frac{\phi^{\frac{1}{1-\gamma}}(t)\ell(x_0)}{\left(\frac{\partial K}{\partial c}(t, A(t,x_0))\right)^{\frac{1}{1-\gamma}}}\right)\right) \mathrm{d}t$$
$$+ (F(x_0)-1)H,$$

for a continuous function $\ell \Longrightarrow$ it attains its maximum at some $x_0^\star$.

Theorem 5

The maximum in $\tilde{U_P}$ is attained for the map

$$p^\star(t,x) = H + \int_{x_0^\star}^{x} g'(y) \left(\frac{\phi(t)^{\frac{1}{\gamma}} \left[g(y)f(y) + g'(y)F(y) - g'(y)\right]^+}{\gamma^{\frac{1-\gamma}{\gamma}} f(y) \frac{\partial K}{\partial c}\left(t, A(t, x_0^\star)\right)} \right)^{\frac{\gamma}{1-\gamma}} \mathrm{d}y.$$

Define then p for any $(t,c) \in [0,T] \times \mathbb{R}_+$ by

$$p(t,c) = \sup_{x \in [0,1]} \left\{ g(x)\phi(t)\frac{c^\gamma}{\gamma} - p^\star(t,x) \right\}.$$

If $p^\star$ is $u-$convex, then p is the optimal tariff for the problem U_P. Furthermore, the Principal only signs contracts with the Agents of type $x \in [x_0^\star, 1]$.

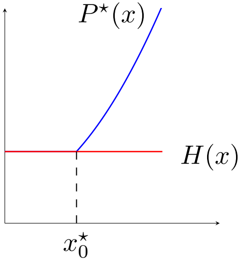

Figure : $X^\star(p^\star) = [x_0^\star, 1]$.

The company prefers the individuals who can pay more.

2) H is not constant. To avoid complex forms of $X^\star$, we assume that

$$\frac{g'(x)}{g(x)} \geq \frac{H'(x)}{H(x)}.$$

Under this assumption we have the following result

Proposition 3

Let $p^\star \in C^+$ be any function such that the set

$$Y^\star(p^\star) := \left\{ x \in [0,1], \int_0^T p^\star(t,x)\mathrm{d}t = H(x) \right\},$$

has positive Lebesgue measure. Then $p^\star$ is not optimal for problem $\tilde{U_P}$.

Thanks to the previous proposition we can redefine C^+ with the additional condition that the measure of $Y^\star(p^\star)$ is zero. For these functions, we define

$$\widehat{X}^\star(p^\star) := X^\star(p^\star) \setminus Y^\star(p^\star) = \{x \in [0,1], P^\star(x) > H(x)\},$$

which is an open subset of $[0,1]$.

$$\Longrightarrow \widehat{X}^\star(p^\star) = [0, b_0) \cup \bigcup_{n \geq 1} (a_n, b_n) \cup (a_0, 1],$$

with $a_0 \in (0,1]$, $b_0 \in [0,1)$, and $b_0 \leq a_{n+1} \leq b_{n+1} \leq a_0$, for every $n \geq 0$. We denote $a := (a_n)_{n \geq 0}$, $b := (b_n)_{n \geq 0}$ and define $\mathcal{A}$ as the set of such that pairs (a,b).

The equivalent formulation we solve is

$$\sup_{(a,b)\in\mathcal{A}} \sup_{p^\star\in C^+(a,b)} \int_0^T \left[\int_{X^\star(a,b)} \left(\frac{g(x)}{g'(x)} \frac{\partial p^\star}{\partial x}(t,x) - p^\star(t,x) \right) f(x)\mathrm{d}x \right.$$
$$\left. - K\left(t, \int_{X^\star(a,b)} \left(\frac{\gamma}{\phi(t)g'(x)} \frac{\partial p^\star}{\partial x}(t,x) \right)^{\frac{1}{\gamma}} f(x)\mathrm{d}x \right) \right] \mathrm{d}t,$$

where

$$C^+(a,b) = \left\{ p^\star \in C^+ : \widehat{X}^\star(p^\star) = [0,b_0) \cup \bigcup_{n\geq 1} (a_n,b_n) \cup (a_0,1] \right\}.$$

By integration this can be re-written as

$$\sup_{(a,b)\in\mathcal{A}} \sup_{p^\star\in C^+(a,b)} \Psi_{(a,b)}(p^\star), \tag{5}$$

with

$$\begin{aligned}
\Psi_{(a,b)}(p^\star) := &\int_0^T \int_0^{b_0} \frac{(g(x)f(x)+g'(x)F(x))}{g'(x)} \frac{\partial p^\star}{\partial x}(t,x)\mathrm{d}x\mathrm{d}t \\
&+ \int_0^T \int_{a_0}^1 \frac{(g(x)f(x)+g'(x)F(x)-g'(x))}{g'(x)} \frac{\partial p^\star}{\partial x}(t,x)\mathrm{d}x\mathrm{d}t \\
&+ \sum_{n=1}^\infty \int_0^T \int_{a_n}^{b_n} \frac{(g(x)f(x)+g'(x)F(x))}{g'(x)} \frac{\partial p^\star}{\partial x}(t,x)\mathrm{d}x\mathrm{d}t \\
&- K\left(t, \int_{X^\star(a,b)} \left(\frac{\gamma}{\phi(t)g'(x)} \frac{\partial p^\star}{\partial x}(t,x)\right)^{\frac{1}{\gamma}} f(x)\mathrm{d}x\right)\mathrm{d}t \\
&+ \sum_{n=1}^\infty F(a_n)H(a_n) - \sum_{n=1}^\infty F(b_n)H(b_n) - F(b_0)H(b_0) + (F(a_0)-1)H(a_0).
\end{aligned}$$

For fixed $(a, b) \in \mathcal{A}$, we consider the problem

$$(P_{a,b}) \qquad \sup_{p^\star \in C^+(a,b)} \Psi_{(a,b)}(p^\star).$$

Since the set $C^+(a, b)$ is very complicated, we need to do local perturbations to obtain optimality conditions providing valuable information.

Theorem 6

Let $p^\star$ be the solution of $(P_{a,b})$. Then for every $x \in X^\star(a,b)$, if $P^\star(x) > H(x)$

$$\frac{\partial p^\star}{\partial x}(t,x) = \begin{cases} \left(\dfrac{\phi(t)^{\frac{1}{\gamma}} \left[g(x)f(x) + g'(x)F(x)\right]^+}{f(x)\frac{\partial K}{\partial c}(t, A(t,a,b))} \right)^{\frac{\gamma}{1-\gamma}} \dfrac{g'(x)}{\gamma}, & \text{if } x \in (0,b_0) \cup \bigcup_n (a_n,b_n), \\ \left(\dfrac{\phi(t)^{\frac{1}{\gamma}} \left[g(x)f(x) + g'(x)F(x) - g'(x)\right]^+}{f(x)\frac{\partial K}{\partial c}(t, A(t,a,b))} \right)^{\frac{\gamma}{1-\gamma}} \dfrac{g'(x)}{\gamma}, & \text{if } x \in (a_0,1), \end{cases}$$

for a continuous function A.

Theorem 6

Let $p^\star$ *be the solution of* $(P_{a,b})$*. Then for every* $x \in X^\star(a,b)$*, if* $P^\star(x) > H(x)$

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for a continuous function A*.*

Since the optimization over $\mathcal{A}$ can be extremely difficult, we assume that the following maps

$$v_1(x) := g'(x) \left(\frac{[g(x)f(x) + g'(x)F(x)]^+}{f(x)} \right)^{\frac{\gamma}{1-\gamma}},$$

$$v_2(x) := g'(x) \left(\frac{[g(x)f(x) + g'(x)F(x) - g'(x)]^+}{f(x)} \right)^{\frac{\gamma}{1-\gamma}},$$

are non-decreasing on $[0, 1]$.

$\Longrightarrow$ the optimal $P^\star$ is convex on intervals where $P^\star > H$.

2.1) H is strictly concave.

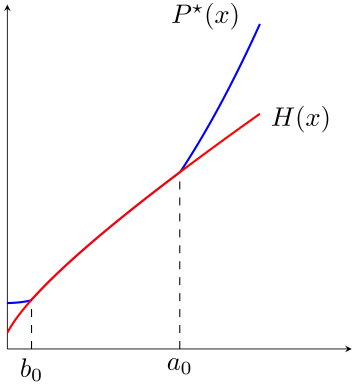


Figure : $\widehat{X}^\star(p^\star)$ for strictly concave H.

$$\Longrightarrow \widehat{X}^\star(p^\star) = [0, b_0) \cup (a_0, 1], \qquad 0 \le b_0 \le a_0 \le 1.$$

Define the set

$$\mathcal{A}_2 := \left\{(a,b) \in [0,1]^2,\ b \leq a\right\}.$$

Problem (5) reduces to

$$\sup_{(a_0,b_0)\in\mathcal{A}_2} \int_0^T \left[\frac{\phi(t)^{\frac{1}{1-\gamma}}\,\ell(a_0,b_0)}{\gamma\left(\frac{\partial K}{\partial c}(t,A(t,a_0,b_0))\right)^{\frac{\gamma}{1-\gamma}}} - K\left(t, \frac{\phi(t)^{\frac{1}{1-\gamma}}\,\ell(a_0,b_0)}{\left(\frac{\partial K}{\partial c}(t,A(t,a_0,b_0))\right)^{\frac{1}{1-\gamma}}}\right)\right] \mathrm{d}t \\ + \theta(a_0,b_0).$$

Where ℓ and θ are continuous on $[0,1]^2$ so the supremum over the compact set above is attained at some $(a_0^\star, b_0^\star) \in \mathcal{A}_2$.

Theorem 7

If the following holds

$$H(b_0^\star) - \frac{\phi(t)^{\frac{1}{1-\gamma}}}{\gamma\left(\frac{\partial K}{\partial c}\left(t, A(t, a_0^\star, b_0^\star)\right)\right)^{\frac{\gamma}{1-\gamma}}} \int_0^{b_0^\star} v_1(y)\mathrm{d}y > H(0),\ t \in [0, T],$$

the maximum in (5) *is attained for the map*

$$p^\star(t,x) = \begin{cases} H(b_0^\star) - \dfrac{\phi(t)^{\frac{1}{1-\gamma}}}{\gamma\left(\frac{\partial K}{\partial c}(t, A(t, a_0^\star, b_0^\star))\right)^{\frac{\gamma}{1-\gamma}}} \displaystyle\int_x^{b_0^\star} v_1(y)\mathrm{d}y, & \text{if } x \in [0, b_0^\star), \\ \tilde{p}^\star(t,x) < H(x), & \text{if } x \in [b_0^\star, a_0^\star], \\ H(a_0^\star) + \dfrac{\phi(t)^{\frac{1}{1-\gamma}}}{\gamma\left(\frac{\partial K}{\partial c}(t, A(t, a_0^\star, b_0^\star))\right)^{\frac{\gamma}{1-\gamma}}} \displaystyle\int_{a_0^\star}^{x} v_2(y)\mathrm{d}y, & \text{if } x \in (a_0^\star, 1]. \end{cases}$$

Theorem 7

Otherwise, it is attained for

$$p^\star(t,x) = \begin{cases} \tilde{p}^\star(t,x) < H(x), \text{ if } x \in [0, a_0^\star], \\ H(a_0^\star) + \dfrac{\phi(t)^{\frac{1}{1-\gamma}}}{\gamma\left(\frac{\partial K}{\partial c}(t, A(t, a_0^\star, b_0^\star))\right)^{\frac{\gamma}{1-\gamma}}} \displaystyle\int_{a_0^\star}^{x} v_2(y)\mathrm{d}y, \text{ if } x \in (a_0^\star, 1]. \end{cases}$$

Define then p*, for any* $(t,c) \in [0,T] \times \mathbb{R}_+$*, by*

$$p(t,c) := \sup_{x\in[0,1]} \left\{ g(x)\phi(t)\frac{c^\gamma}{\gamma} - p^\star(t,x) \right\}.$$

If $p^\star$ *is* $u-$*convex, then* p *is the optimal tariff for the problem* U_P*. Furthermore, the Principal only signs contracts with the Agents of type* $x \in [0, b_0^\star] \cup [a_0^\star, 1]$ *in the first case and with the Agents of type* $x \in [a_0^\star, 1]$ *in the second case.*

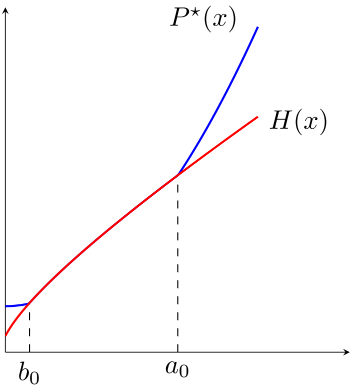


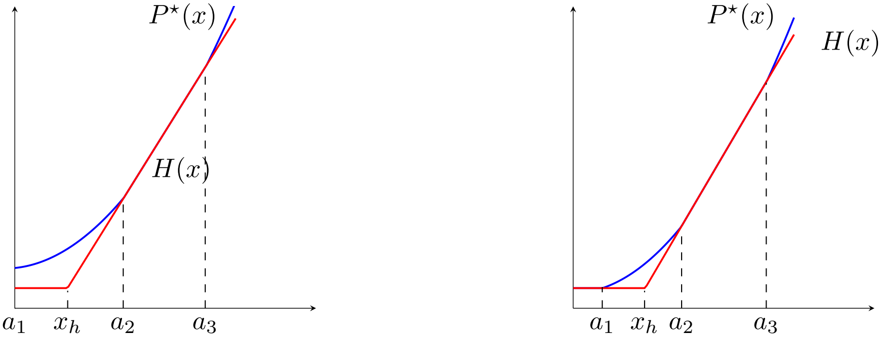
Figure : $\widehat{X}^\star(p^\star)$ for strictly concave H.

The company prefers the individuals who can pay more and the ones who are not so difficult to satisfy.

2.2) H is constant-linear.

$$H(x) = \begin{cases} \beta, \text{ if } x \in [0, x_h], \\ \alpha(x - x_h) + \beta, \text{ if } x \in [x_h, 1], \end{cases}$$

where $\alpha, \beta \geq 0$ and where $x_h \in [0, 1]$.



(a) $a_1 = 0$. (b) $a_1 > 0$.

Figure : $X^\star(p^\star)$ for a "constant-linear" H.

$$\Longrightarrow X^\star(p^\star) = [a_1, a_2] \cup [a_3, 1], \qquad 0 \le a_1 \le x_h \le a_2 \le a_3 \le 1.$$

Let us then define the set

$$\mathcal{A}_3 := \left\{(a,b,c) \in [0,1]^3,\ a \le x_h \le b \le c\right\}.$$

Problem (5) becomes

$$\sup_{(a_1,a_2,a_3)\in\mathcal{A}_3} \int_0^T \left[\frac{\phi(t)^{\frac{1}{1-\gamma}} \ell(a_1,a_2,a_3)}{\gamma \left(\frac{\partial K}{\partial c}(t, A(t,a_1,a_2,a_3))\right)^{\frac{\gamma}{1-\gamma}}} - K\left(t, \frac{\phi(t)^{\frac{1}{1-\gamma}} \ell(a_1,a_2,a_3)}{\left(\frac{\partial K}{\partial c}(t, A(t,a_1,a_2,a_3))\right)^{\frac{1}{1-\gamma}}}\right)\right] \mathrm{d}t + \theta(a_1,a_2,a_3),$$

for some continuous maps A, θ and ℓ.

Merci de votre attention!

Appendix.

For every $(t, x_0) \in [0, T] \times [0, 1]$

$$A(t, x_0) = g_K^{(-1)} \left(\phi^{\frac{1}{1-\gamma}}(t) \int_{x_0}^1 \left(\frac{\left[g_\gamma(x) f(x) + g'_\gamma(x) F(x) - g'_\gamma(x)\right]^+}{f^\gamma(x)} \right)^{\frac{1}{1-\gamma}} \mathrm{d}x \right),$$

$$g_K(c) := c \left(\frac{\partial K}{\partial c}(t, c) \right)^{\frac{1}{1-\gamma}}, \; c \geq 0,$$

and

$$\ell(x_0) := \int_{x_0}^1 \left(\frac{[g(x) f(x) + g'(x) F(x) - g'(x)]^+}{f^\gamma(x)} \right)^{\frac{1}{1-\gamma}} \mathrm{d}x.$$

For any $(t, a_0, b_0) \in [0, T] \times \mathcal{A}_2$

$$A(t, a_0, b_0) := g_K^{(-1)} \left(\phi(t)^{\frac{1}{1-\gamma}} \int_0^{b_0} \left(\frac{[g(x)f(x) + g'(x)F(x)]^+}{f^\gamma(x)} \right)^{\frac{1}{1-\gamma}} \mathrm{d}x \right.$$

$$\left. + \phi(t)^{\frac{1}{1-\gamma}} \int_{a_0}^{1} \left(\frac{[g(x)f(x) + g'(x)F(x) - g'(x)]^+}{f^\gamma(x)} \right)^{\frac{1}{1-\gamma}} \mathrm{d}x \right),$$

$$\ell(a_0, b_0) := \int_0^{b_0} \left(\frac{[g(x)f(x) + g'(x)F(x)]^+}{f^\gamma(x)} \right)^{\frac{1}{1-\gamma}} \mathrm{d}x$$

$$+ \int_{a_0}^{1} \left(\frac{[g(x)f(x) + g'(x)F(x) - g'(x)]^+}{f^\gamma(x)} \right)^{\frac{1}{1-\gamma}} \mathrm{d}x,$$

$$\theta(a_0, b_0) := -F(b_0)H(b_0) + (F(a_0) - 1)H(a_0).$$

For any $(t, a_1, a_2, a_3) \in [0,T] \times \mathcal{A}_3$

$$\begin{aligned}
\ell(a_1,a_2,a_3) &:= \int_{a_1}^{a_2} \left(\frac{[g(x)f(x) + g'(x)F(x)]^+}{f^\gamma(x)} \right)^{\frac{1}{1-\gamma}} \mathrm{d}x \\
&\quad + \int_{a_3}^{1} \left(\frac{[g(x)f(x) + g'(x)F(x) - g'(x)]^+}{f^\gamma(x)} \right)^{\frac{1}{1-\gamma}} \mathrm{d}x, \\
\theta(a_1,a_2,a_3) &:= F(a_1)H(a_1) - F(a_2)H(a_2) + (F(a_3)-1)H(a_3), \\
A(t,a_1,a_2,a_3) &:= g_K^{(-1)} \left(\phi(t)^{\frac{1}{1-\gamma}} \int_{a_1}^{a_2} \left(\frac{[g(x)f(x) + g'(x)F(x)]^+}{f^\gamma(x)} \right)^{\frac{1}{1-\gamma}} \mathrm{d}x \right. \\
&\quad \left. + \phi(t)^{\frac{1}{1-\gamma}} \int_{a_3}^{1} \left(\frac{[g(x)f(x) + g'(x)F(x) - g'(x)]^+}{f^\gamma(x)} \right)^{\frac{1}{1-\gamma}} \mathrm{d}x \right).
\end{aligned}$$

Theorem 8

Let $f(x) = 1$, $g(x) = x$ and the cost function K be given, for some $n > 1$, by

$$K(t,c) := k(t)\frac{c^n}{n}.$$

Then, the optimal tariff $p \in \mathcal{P}$ is given for any $(t,c) \in [0,T] \times \mathbb{R}_+$ by

$$p(t,c) = \begin{cases} \phi(t)\dfrac{c^\gamma}{\gamma} + M(t)\left((2x_0^\star - 1)^{\frac{1}{1-\gamma}} - 1\right) - h(t), \text{ if } c > \left(\dfrac{2\gamma M(t)}{(1-\gamma)\phi(t)}\right)^{\frac{1}{\gamma}}, \\ \phi(t)\dfrac{c^\gamma}{2\gamma} + c\left(\left(\dfrac{\phi(t)}{2}\right)^{\frac{1}{1-\gamma}} \dfrac{1-\gamma}{\gamma M(t)}\right)^{\frac{1-\gamma}{\gamma}} - h(t) + M(t)(2x_0^\star - 1)^{\frac{1}{1-\gamma}}, \text{ if not,} \end{cases}$$

where $M(t) = \dfrac{1-\gamma}{2\gamma}\left(\dfrac{2(2-\gamma)}{1-\gamma}\right)^{\frac{\gamma(n-1)}{n-\gamma}}\left(\dfrac{\phi^n(t)}{k^\gamma(t)}\right)^{\frac{1}{n-\gamma}}\left(1 - (2x_0^\star - 1)^{\frac{2-\gamma}{1-\gamma}}\right)^{-\frac{\gamma(n-1)}{n-\gamma}}$,

and where $x_0^\star$ is the unique solution in $(1/2, 1)$ of the equation

$$\int_0^T h(t)\mathrm{d}t = 2nA(T)\frac{2-\gamma}{n-\gamma}(2x_0^\star - 1)^{\frac{1}{1-\gamma}}\left(1 - (2x_0^\star - 1)^{\frac{2-\gamma}{1-\gamma}}\right)^{-\frac{\gamma(n-1)}{n-\gamma}}.$$

Furthermore, only the Agents of type $x \geq x_0^\star$ will accept the contract.