# Viscosity Solutions of Path-Dependent PDEs

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# PDE characterization : linear exmaple

## Linear Expectation

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Function $v$ is $C^{1,2}$, and is a classical solution of the heat equation.

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In the linear case, the martingale characterization as an alternative gives quite a lot analytic insight, and can be naturally generalized to the non-Markovian case.

# PDE characterization : beyond the linear case

Consider a controlled diffusion:

$$X_t^\kappa = X_0 + \int_0^t b(s, X_s^\kappa, \kappa_s)ds + \int_0^t \sigma(s, X_s^\kappa, \kappa_s)dW_s$$

for $\kappa \in \mathcal{K} = \{\kappa : \kappa_t \in K \text{ for all } t \in [0, T]\}$.

**Value function of optimal control**

$$v(t,x) = \sup_{\kappa\in\mathcal{K}} \mathbb{E}\Big[ \int_t^T f(s, X_s^\kappa, \kappa_s)ds + h(X_T^\kappa) \Big| X_t^\kappa = x \Big]$$

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**Hamilton-Jacobi-Bellman Equation**

$$\partial_t u + \sup_{k \in K} \Big\{ b \cdot Du + \frac{1}{2} \mathrm{Tr}\big( (\sigma\sigma^{\mathrm{T}}) D^2 u \big) + f \Big\} = 0, \quad u(T,x) = h(x).$$

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### PDE characterization (under some conditions)

Function $v$ is a *viscosity solution* of the HJB equation.

# Non-Markovian model

Consider the diffusion $X$ controlled with delay:
$X_t^\kappa = X_0 + \int_0^t b(s, X_{s-\delta}^\kappa, \kappa_s)ds + \int_0^t \sigma(s, X_{s-\delta}^\kappa, \kappa_s)dW_s, \quad \kappa \in \mathcal{K}$

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$$v_t = \sup_{\kappa\in\mathcal{K}} \mathbb{E}\Big[ \int_t^T f(s, X_{s-\delta}^\kappa, \kappa_s)ds + h(X_T^\kappa)\Big|\mathcal{F}_t\Big]$$

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It is IMPOSSIBLE to find a corresponding PDE of finite dimension state space !

# A first meeting with Path-dependent PDE (PPDE)

**Linear Expectation: non-Markovian**

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Introduce *viscosity solutions* to PPDE's

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# 'The' unique well-defined solution

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Only one continuous solution fits the maximum principle: $u(x) = |x|$.

# Why 'the' unique solution?

Add a perturbation to the previous equation:

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However, the maximum principle as a criteria is NOT easy to verify a priori. It is more like a property instead of a definition of solutions.

# Wait... The simple example can tell more...

Consider the perturbation with negative Laplacian:

$$-|Dv^\varepsilon(x)| + \varepsilon \Delta v^\varepsilon = -1,\ x \in (-1,1), \quad v^\varepsilon(-1) = v^\varepsilon(1) = 1$$

The solutions are $v^\varepsilon(x) = 2 - u^\varepsilon(x)$ converging to $2 - u(x)$.

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How can it be true ?!

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Split the eq. to one sub-equation and one super-equation

$$-|Du| \leq, \geq -1 \quad \text{and} \quad |Dv| \leq, \geq 1$$

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A good definition of viscosity solution should treat the sub-eq. and the super-eq. separately.

# Test functions of viscosity solutions (heat equation)

Consider the heat equation : $-\mathcal{L}u := -(\partial_t u + \frac{1}{2}\Delta u) = 0$, $u(T, \cdot) = g$.

To define a weak solution, first define the *test functions*.

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To define a weak solution, first define the *test functions*. Consider all the smooth functions tangent to $u$ from above at point $(t,x)$, namely,

$$\underline{A}u(t,x) \ := \ \{\varphi \in C^{1,2} : 0 = (u-\varphi)(t,x) = \max_{s,y}(u-\varphi)(s,y)\}$$

$$-(\partial_t\varphi + \tfrac{1}{2}\Delta\varphi)(t,x) \leq 0 \text{ for all } \varphi \in \underline{A}u(t,x).$$

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Let $W$ be a Brownian motion. As a solution of the heat eq., $\{u(t+s, x+W_s)\}_s$ is naturally a martingale. Therefore, we have

$$-\varphi(t,x) \geq \mathbb{E}[(u-\varphi)(t+\tau, x+W_\tau) - u(t,x)] = \mathbb{E}[-\varphi(t+\tau, x+W_\tau)], \forall \tau$$

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# Definition of viscosity solutions (heat equation)

Based on the previous observation, we may guess a definition for the viscosity solution of the heat eq.

## Definition (Viscosity solution of heat eq.)

*Function u is continuous.*

- *u is a viscosity sub-solution if* $-\mathcal{L}\varphi(t,x) \le 0, \forall t,x, \ \varphi \in \underline{A}u(t,x)$
- *v is a viscosity super-solution if* $-\mathcal{L}\varphi(t,x) \ge 0, \forall t,x, \ \varphi \in \overline{A}v(t,x)$
- *u is a viscosity solution if u is both visco. sub- and super-solution.*

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Based on the previous observation, we may guess a definition for the viscosity solution of the heat eq. Let $\mathbb{P}_0$ be the Wiener's measure.

**Definition ($\mathbb{P}_0$-viscosity solution of heat eq.)**

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(See [Bayraktar, Sirbu 2012], [Ekren, Keller, Touzi, Zhang 2014])

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- *u is a $\mathbb{P}_0$-visco. solution if u is both $\mathbb{P}_0$-visco. sub- and super-solution.*

(See [Bayraktar, Sirbu 2012], [Ekren, Keller, Touzi, Zhang 2014])

Is it a good definition ?

- Is there a unique solution?
- Does it satisfy the maximum principle?

# Two puzzles merge into one : *Comparison Principle*

## Comparison principle

Let $u, v$ be ($\mathbb{P}_0$-)viscosity sub-/super-solution, respectively. Given the fact $u(T, \cdot) \leq v(T, \cdot)$, then we have $u \leq v$ everywhere.

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• The comparison principle directly leads to the uniqueness of the ($\mathbb{P}_0$-)viscosity solutions to the Dirichlet problem.

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• Take the constant function $v \equiv \max_y u(T,y)$. Then $v$ is a (super)solution to the heat equation and $u(T,\cdot) \leq v$. By the comparison principle, we obtain $u(\cdot,\cdot) \leq v$,

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• Take the constant function $v \equiv \max_y u(T,y)$. Then $v$ is a (super)solution to the heat equation and $u(T,\cdot) \leq v$. By the comparison principle, we obtain $u(\cdot,\cdot) \leq v$, i.e.

**Maximum principle (Parabolic)**

Let $u$ be ($\mathbb{P}_0$-)viscosity solution. We have $\max_{t\leq T,x} u(t,x) = \max_x u(T,x)$.

# Proof of comparison for $\mathbb{P}_0$-viscosity solutions

By an optimal stopping argument, we may easily prove:

**Theorem**

*Under some integrability condition, the following properties are equivalent:*

- *$u$ is a $\mathbb{P}_0$-visco.super-(sub-)solution to the heat equation;*
- *$u(t, W_t)$ is a super-(sub-)martingale.*

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<ins>Proof of comparison</ins> : Let $u, v$ be $\mathbb{P}_0$-visco-sub/super-solution respectively and assume that $u(T, \cdot) \le v(T, \cdot)$. Since $u(t, W_t)$ is a submartingale and $v(t, W_t)$ is a supermartingale, we have

$$u(t,x) \le \mathbb{E}[u(T, W_T)|W_t = x] \le \mathbb{E}[v(T, W_T)|W_t = x] \le v(t,x)$$

for all $t, x$. □

# Why we prefer the $\mathbb{P}_0$-viscosity solution definition?

By considering the test functions tangent in mean value instead of those tangent point-wisely, we have more test functions, and so fewer visco-sub- or super-solutions. Intuitively, it helps to prove the comparison principle.

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Real space is locally compact

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$\exists\ x = \mathrm{argmax}_{y \in O} u(y)$

**Path dependent PDE**

Path space is NOT

# Why we prefer the $\mathbb{P}_0$-viscosity solution definition?

By considering the test functions tangent in mean value instead of those tangent point-wisely, we have more test functions, and so fewer visco-sub- or super-solutions. Intuitively, it helps to prove the comparison principle.

Technically, by considering the test functions tangent in mean value, we overcome the following difficulty:

**PDE**

Real space is locally compact

$\exists\ x = \mathrm{arg}\max_{y \in O} u(y)$

**Path dependent PDE**

Path space is NOT

$\exists\ \tau^* = \mathrm{arg}\max_{\tau} \mathbb{E}^{\mathbb{P}_0}[u_\tau]$

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  replace the smooth test functions on the real space by the ones on the path space (Dupire derivatives), or just consider the paraboloids $\varphi^{a,b,c}(t,\omega) = at + b \cdot \omega_t + \frac{1}{2}\omega_t^T c\omega_t$ as the test functions

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- Extension to the nonlinear equations (i.e. $-\partial_t u - G(t,\omega,u,\partial_\omega u,\partial^2_{\omega\omega} u) = 0$)

  replace the linear expectation $\mathbb{E}^{\mathbb{P}_0}$ by the nonlinear ones $\overline{\mathcal{E}}^{\mathcal{P}}$ or $\underline{\mathcal{E}}^{\mathcal{P}}$, where $\overline{\mathcal{E}}^{\mathcal{P}} := \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}, \underline{\mathcal{E}}^{\mathcal{P}} := \inf_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}$, and $\mathcal{P}$ is a family of continuous semi-martingale measures.

  We can prove comparison results under appropriate conditions on $G$.

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# How much the delay matters?

Consider the control problems with delays corresponding to the PPDE

$$-\partial_t u^\delta - G(t, \omega_{t-\delta}, \partial^2_{\omega\omega} u^\delta) = 0, \quad u^\delta(T, \omega_T) = h(\omega_T)$$

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Define $H_t^\delta := \frac{1}{\delta}\mathbb{E}\big[\int_t^T \big(G(t, X_{t-\delta}, u(t, X_t)) - G(t, X_t, u(t, X_t))\big)\big|\mathcal{F}_t\big]$, where $dX_t = (2G_\gamma(t, X_t, u(t, X_t)))^{\frac{1}{2}} dW_t$

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Under appropriate conditions (regularity of $G, u$), we may prove that $\lim_{\delta\to 0} \frac{u^\delta - u}{\delta} = \lim_{\delta\to 0} H^\delta$ and the r.h.s. can be calculated explicitly.

# Intuitive proof

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&= -\frac{1}{\delta}\big(G(t,\omega_{t-\delta},D^2u) - G(t,\omega_{t-\delta},\partial^2_{\omega\omega}u^\delta)\big) \\
&\qquad\qquad + \frac{1}{\delta}\big(G(t,\omega_t,D^2u) - G(t,\omega_{t-\delta},D^2u)\big) - \partial_t H^\delta \\
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\end{aligned}
$$

By stability argument, we may prove $\partial_t v + G_\gamma(t,\omega_t,D^2u)\partial^2_{\omega\omega}v = 0$. Taking into account that $v_T = 0$, we obtain $v \equiv 0$.

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# Thank you for your attention!