Solving Quadratic BSDEs

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What is a solution to a BSDE ?

For BSDE(ξ, g) : a pair (Y, Z) of predictable processes such that a.s $t \mapsto Y_t$ is continuous , $t \mapsto Z_t$ belongs to $L^2(0, T)$, $t \mapsto g(t, Y_t, Z_t)$ belongs to $L^1(0, T)$ and

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)\, ds - \int_t^T Z_s dWs$$

where $\{W_t := (W_t^1, ..., W_t^d)^*, 0 \leq t \leq T\}$ is a d-dimensional standard Brownian motion defined on some probability space $(\Omega, \mathscr{F}, P)$

Notations and requirements

n denotes the dimension for Y

$\diamond \{\mathscr{F}_t, 0 \leq t \leq T\}$: augmented natural filtration of the Brownian motion W

$\diamond \mathscr{S}^\infty(\mathbb{R}^n)$: set of $\mathbb{R}^n$-valued $\mathscr{F}_t$-adapted essentially bounded continuous processes. Banach space when provided with the essential sup norm $||.||_\infty$

$\diamond \mathscr{M}^2(\mathbb{R}^{n\times d})$: set of predictable $\mathbb{R}^{n\times d}$-valued processes $\{Z_t\}_{t\in[0,T]}$ such that $||Z||_{\mathscr{M}^2} := \mathbb{E}\left[\int_0^T |Z_s|^2\, ds\right]^{1/2} < \infty$

- Comparison theorem and Kobylanski's monotone convergence theorem

$\diamond \mathscr{E}(M)$: the stochastic exponential of a one-dimensional local martingale M

$\diamond \beta \cdot M$: the stochastic integral of an adapted process β with respect to a local continuous martingale M

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• Girsanov's theorem and *BMO* martingales

Definition : $M = (M_t, \mathscr{F}_t)$ uniformly integrable martingale with $M_0 = 0$ is BMO_2 if there exists $c > 0$ so that $\mathbb{E}[< M >_\tau^\infty | \mathscr{F}_\tau] \leq c$ for all bounded s.t τ.
$||M||^2_{BMO_2} :=$ the smallest such constant.

Lemma (Kazamaki) :

For $K > 0$, there are constants $c_1(K) > 0$ and $c_2(K) > 0$ such that for any one-dimensional BMO_2 martingale N such that $||N||_{BMO_2(P)} \leq K$ and any BMO_2 martingale M,

$$c_1||M||_{BMO_2(P)} \leq ||\tilde{M}||_{BMO_2(\tilde{P})} \leq c_2||M||_{BMO_2(P)}$$

where $\tilde{M} := M - \langle M, N\rangle$ and $d\tilde{P} := \mathscr{E}(N)_0^\infty dP$.

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- Existence and uniqueness of solution to BSDE with quadratic generator

Theorem (Briand, Hu - 2006) :

If $|g(t,y,z)| \leq \alpha_t + \beta|y| + \frac{\gamma}{2}|z|^2$ with $(\alpha_t)_{t\in[0,T]}$ progressively measurable and $\exists\lambda > \gamma e^{\beta T}$ s.t $\mathbb{E}[exp(\lambda|\xi| + \lambda\int_0^T \alpha_t\, dt)] < \infty$ then their exists a solution with $Z \in \mathscr{M}^2$ and $-\frac{1}{\gamma}\, ln\mathbb{E}[\phi_t(-\xi)|\mathcal{F}_t] \leq Y_t \leq \frac{1}{\gamma}\, ln\mathbb{E}[\phi_t(\xi)|\mathcal{F}_t]$

where $\phi_.(z)$ solves $\begin{cases} d\phi_t = -H(t,\phi_t)dt \\ \phi_T = e^{\gamma z} \end{cases}$ with $H(t,p) := \alpha_t\gamma\mathbf{1}_{p\in]-\infty,1[} + p(\alpha_t\gamma + \beta ln(p))\,\mathbf{1}_{p\geq 1}$

Theorem (Delbaen, Hu, Richou - 2011) :

Suppose g convex with respect to z, $K-$Lip with respect to y and

$$g(t,y,z) \leq \bar{\alpha}_t + \bar{\beta}|y| + \frac{\bar{\gamma}}{2}|z|^2.$$

1) Suppose $g(t,y,z) \geq -\underline{\alpha}_t - r(|y| + |z|)$. If exponential moment of order ε for $\xi^- + \int_0^T \underline{\alpha}_t\, dt$ then a solution and a cste C s.t $\mathbb{E}[e^{\frac{\varepsilon}{C}e^{-CT}(Y^-)^*}] < \infty$

If exponential moment of order $pe^{\bar{\beta}T}, p > \bar{\gamma}$ for $\xi^+ + \int_0^T \bar{\alpha}_t\, dt$ then a solution s.t $\mathbb{E}[e^{pA^*}] < \infty$ with $A_t := Y_t^+ + \int_0^T \bar{\alpha}_t\, dt$

2) If there exists a solution verifying $\exists p > \bar{\gamma}, \exists\varepsilon > 0$ s.t $\mathbb{E}[e^{pA^*} + e^{\varepsilon(Y^-)^*}] < \infty$ then it is unique (among such solutions).

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Motivation

Zero-sum stochastic differential games (Hamadene, Lepeltier - 1995) :

Equation $\begin{cases} dY_t = g(Z_t)dt - Z_t.dW_t \\ Y_T = \xi \\ g(z) = g_1(z) - g_2(z) = \sup_{q_1 \in U} \inf_{q_2 \in V} \{z.(q_1 - q_2) - (f_1(q_1) - f_2(q_2))\} \end{cases}$

written as $\begin{cases} -dY_t = \tilde{g}(\tilde{Z}_t)dt - \tilde{Z}_t.dW_t \\ Y_T = \xi \\ \tilde{g}(z) = \inf_{q_1 \in U} \sup_{q_2 \in V} \{z.(q_1 - q_2) + (f_1(q_1) - f_2(q_2))\} \end{cases}$

admits a unique solution : the payoff with optimal strategies

$$Y_t = J_t(\tilde{u}^*, \tilde{v}^*) = \inf_{u \in \mathcal{U}} \sup_{v \in \mathcal{V}} J_t(u, v) \quad \text{where}$$

$$J_t(u,v) = \mathbb{E}^{u,v}\left[\xi + \int_t^T (f_1(u_s) - f_2(v_s))ds | \mathscr{F}_t\right], \; \frac{d\mathbb{P}^{u,v}}{d\mathbb{P}} = \mathcal{E}\left(\int (u - v).dW\right)$$

The Legendre-Fenchel transformation

Proposition :

The Legendre-Fenchel transformation of g defined on $[0, T] \times \mathbb{R} \times \mathbb{R}^d$ by

$$f(t, y, q) := \sup\{z.q - g(t, y, z) \,|\, z \in \mathbb{R}^d\}$$

is convex $/q$, $K-$Lip $/y$ (when finite) and $f(t, y, q) \geq -\bar{\alpha}_t - \bar{\beta}|y| + \frac{1}{2\bar{\gamma}}|q|^2$.

Geometrical interpretation :

Sub-differential of $g(t, y, .)$ at point z_0 (denoted $\partial g(t, y, .)(z_0)$)
$:=$ set of the slopes of all affine functions $\leq g(t, y, .)$ but equal in z_0.

$$(\, q.z - c \leq g(t, y, z) \;\forall z \,) \Leftrightarrow (\, c \geq q.z - g(t, y, z) \;\forall z \,) \Leftrightarrow c \geq f(t, y, q)$$

If sup reached in z_0, then $q \in \partial g(t, y, .)(z_0)$.

The result

$$dY_t = [g_1(Y_t, Z_t) - g_2(Y_t, Z_t)]\, dt - Z_t.dW_t \quad , \quad Y_T = \xi$$

Assumptions : g_i convex $/z$, $K-$Lip $/y$, $0 \leq g_i(y,z) \leq C + \beta|y| + \frac{1}{2}|z|^2$
exponential moment of order $> e^{(\beta+K)T}$ for $|\xi|$

Then their exists (Y, Z) solution with $Z \in \mathscr{M}^2$

Theorem : ***H.***

If g_2 $L-$Lip $/z$, $f_2(0,.)$ bounded on its effective domain and $\exists p > 1$ and $\varepsilon > 0$ s.t $\mathbb{E}[e^{p(Y^-)^*} + e^{\varepsilon(Y^+)^*}] < \infty$ then uniqueness and characterization :

$$Y_t = \sup_{q \in \mathcal{E}^{[0,T]}} Y_t^q \quad \begin{cases} \mathcal{E} := \{q \in \mathbb{R}^d \mid |q| \leq L, f_2(0,q) < \infty\} \\ dY_t^q = [g_1(Y_t^q, Z_t^q) - Z_t^q.q_t + f_2(Y_t^q, q_t)]\, dt - Z_t^q.dW_t \end{cases}$$

the supremum is reached in any element of the sub-differential of $g_2(Y,.)$ at Z.

Framework

$$\begin{cases} X_s^{t,x} = x & \forall s \leq t \\ (X_s^{t,x})_{s \geq t} & \text{unique solution of SDE} \end{cases}$$

$$X_s^{t,x} = x + \int_t^s b(u, X_u^{t,x})du + \int_t^s \sigma(u)dW_u \qquad \forall s \geq t$$

where $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : [0,T] \to \mathbb{R}^{d\times d}$ continuous, $|b(.,0)| \leq K$ and b K-Lipschitz w.r.t space variable.

$$Y_s^{t,x} = \varphi(X_T^{t,x}) + \int_s^T g(Y_u^{t,x}, Z_u^{t,x})du - \int_s^T Z_u^{t,x} dW_u \qquad \forall s \in [0,T] \quad (*_1)$$

Assumptions :

- φ is bounded
- $g^i(y,z) = f^i(z^i) + h^i(y,z)$
- $|f^i(z^i)| \leq C + \frac{\gamma}{2}|z^i|^2, \qquad |f^i(z^i) - f^i(\tilde{z}^i)| \leq C(1 + |z^i| + |\tilde{z}^i|)|z^i - \tilde{z}^i|$
- $|h(y,z)| \leq C(1+|y|)\ , \quad |h(y,z) - h(\tilde{y},\tilde{z})| \leq C(|y-\tilde{y}| + |z-\tilde{z}|)$

Local $\rightarrow$ Global solution

Proposition (Hu,Tang - 2016) :

There exists $\varepsilon > 0$ and a bounded set $\mathscr{B}_\varepsilon$ of the product space $\mathscr{S}^\infty \times BMO_2$ restricted on the time interval $[T-\varepsilon, T]$ such that BSDE $(*_1)$ has a unique local solution $(Y^{t,x}, Z^{t,x})$ in the time interval $[T-\varepsilon, T]$ with $(Y, Z) \in \mathscr{B}_\varepsilon$.

$\Pi : (U, V \cdot W) \in \mathscr{S}^\infty \times BMO_2 \longmapsto (Y, Z \cdot W) \in \mathscr{S}^\infty \times BMO_2$ with coordinates solution to the n equations

$$Y_s^i = \varphi^i(X_T^{t,x}) + \int_s^T [f^i(Z_u^i) + h^i(U_u, V_u)]\, du - \int_s^T Z_u^i dW_u \qquad \forall s \in [0, T]$$

$$\forall \varepsilon \leq (3nC)^{-1}, A_\varepsilon := \frac{2n}{\gamma^2} e^{\gamma ||\varphi(X_T^{t,x})||_\infty} + 2Cn\varepsilon(1+\tfrac{1}{\gamma})(1+\tfrac{1}{\gamma^2}) e^{6\gamma CT} e^{3n\gamma ||\varphi(X_T^{t,x})||_\infty}$$

$$\mathscr{B}_\varepsilon := \left\{ (U, V) \,\middle|\, e^{2\gamma ||U||_{\infty,\varepsilon}} \leq e^{6\gamma CT} e^{3n\gamma ||\varphi(X_T^{t,x})||_\infty},\ ||V \cdot W||^2_{BMO_2,\varepsilon} \leq A_\varepsilon \right\}$$

$K^2 := 3C^2T + 6C^2A_{(3nC)^{-1}}$, $(Y, Z) := \Pi(U, V)$ and $(\tilde{Y}, \tilde{Z}) := \Pi(\tilde{U}, \tilde{V})$

$$||Y - \tilde{Y}||^2_{\infty,\varepsilon} + c_1(K)^2||(Z - \tilde{Z}) \cdot W||^2_{BMO_2,\varepsilon}$$
$$\leq 4nC^2\varepsilon^2||U - \tilde{U}||^2_{\infty,\varepsilon} + 4nC^2\varepsilon c_2(K)^2||(V - \tilde{V}) \cdot W||^2_{BMO_2,\varepsilon}$$

$\bar{K} \leftarrow$ replace $||\varphi(X_T^{t,x})||_\infty$ by $\sqrt{\lambda}$ with $\lambda \geq ||\varphi(X_T^{t,x})||^2_\infty$ s.t $||Y^{t,x}||^2_{\infty,\varepsilon} \leq \lambda$

η_λ s.t $4nC^2\eta_\lambda \max\{\eta_\lambda, \left[\frac{c_2(\bar{K})}{c_1(\bar{K})}\right]^2\} < 1$ permits to **iterate**

Theorem (Hu,Tang - 2016) :

BSDE $(*_1)$ has a unique global solution $(Y^{t,x}, Z^{t,x})$ on $[0, T]$ such that $Y^{t,x}$ is bounded. Furthermore, $Z \cdot W$ is a BMO_2 martingale.

Stability results

$$u : (t,x) \in [0,T] \times \mathbb{R}^d \mapsto Y_t^{t,x}$$

Proposition : ***H.***

Suppose φ uniformly continuous, then $\forall\, t \in [0,T]$, $u(t,.)$ uniformly continuous.

Theorem : ***H.***

Suppose that b and φ differentiable with respect to the space variable with bounded and uniformly continuous differentials.
Suppose also to have good properties for the partial differentials of h^i and ∇f^i, then u is of class $\mathcal{C}^1$ with respect to the space variable.

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Thank you for your attention