Order book modeling and market making under uncertainty.

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Outline

Electronic trading

- Today a large proportion of transactions in equity markets are executed by algorithms on electronic trading platforms
- In electronic order-driven markets, participants may submit limit orders (or cancel an existing limit order), by specifying whether they wish to buy or sell, the amount (volume) and the price.
- Limit orders wait in a queue to be canceled or executed and the latter occurs when a sell/buy order is matched against one or more buy/sell limit orders.
- All outstanding limit orders are aggregated in a limit order book which is available to market participants.
- The order book at a given instant of time is the list of all outstanding buy and sell limit orders with their corresponding price and volume

orderbook

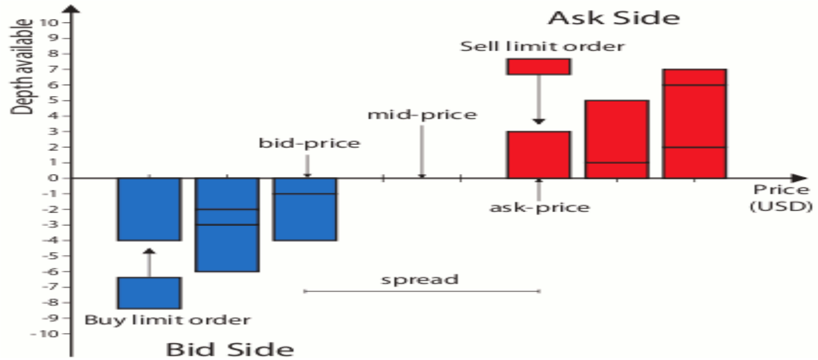

Figure: A snapshot of the limit order book taken at a fixed instance in time.

Point process

The most intuitive models for the order book dynamics are those based on self-exciting point processes. For example

- Let $(t_i)_{i=1,2,\dots}$ be the sequence of times at which new events occur at the order book
- Let $(Z_i)_{i\geq 1}$ be a sequence of random vectors corresponding to the characteristics associated to $(t_i)_{i\geq 1}$
- $(Z_i)_{i\geq 1}$ is called the sequence of marks while the double sequence $(t_i, Z_i)_{i\geq 1}$ is called a simple marked point process.
- We introduce the counting processes corresponding to t_i

$$N(t) := \sum_{i\geq 1} 1_{t_i \leq t},\ \tilde{N} := \sum_{i\geq 1} 1_{t_i < t}$$

  $N(t)$ is right continues with upward jumps at t_i; $\tilde{N}(t)$ is left continuous. $\tilde{N}(t)$ counts the number of event that occurred before t.
- Let $X_i := t_i - t_{i-1}$ be the duration process associated with $(t_i)_{i\geq 1}$. The left continuous process $X_t = t - t_{\tilde{N}(t)}$ is called the backward recurrence time at t.

Point process

- Assuming that $(t_i)_{i\geq 1}$ is generated through a Poisson process with intensity $\lambda(t) = \lambda > 0$ it follows that the waiting time until the first event is exponentially distributed.
- Assume in the following that $N(t)$ is a simple point process on $[0,\infty[$ adapted to some history $\mathcal{F}_t$. If

  $$\lambda(t,\mathcal{F}_t) := \lim_{\Delta\to 0}\frac{1}{\Delta}\mathbb{E}[N(t+\Delta)-N(t)|\mathcal{F}_t],\ \lambda(t,\mathcal{F}_t) > 0 \text{ for all } t > 0,$$

  then $\lambda(t,\mathcal{F}_t)$ is called the $\mathcal{F}_t$-intensity process of the counting process $N(t)$.
- Typically one considers the case $\mathcal{F}_t = \mathcal{F}_t^N$ where $\mathcal{F}_t^N$ consists of the complete observable history of the point process up to t:

  $$\mathcal{F}_t^N = \sigma(t_{N(t)}, Z_{N(t)}, \ldots, t_1, Z_1)$$

- We have

  $$\mathbb{E}[N(t)-N(t')] = \mathbb{E}[\int_{t'}^{t}\lambda(s)ds|\mathcal{F}_t]$$

Point process

- The integrated intensity function

$$\Delta(t_{i-1}, t_i) = \int_{t_{i-1}}^{t_1} \lambda(s, \mathcal{F}_s) ds$$

$\Delta(t_{i-1}, t_i)$ establishes the link between the intensity function and the duration until the occurrence of the next point.

- A statistical model can be completely specified in terms of the $\mathcal{F}_t$-intensity and a likelihood function can be established in terms of the intensity:

$$\log L(W, \theta) = \int_0^{\hat{t}_n} (1 - \lambda(s, \mathcal{F}_s)) ds + \sum_{i \geq 1} 1_{[0, \hat{t}_n]}(\hat{t}_i) \lambda(\hat{t}_i, \mathcal{F}_{\hat{t}_i})$$

Point process

- Consider a simple marked point process $(t_i, Z_i)_{i\geq 1}$ where the basic self-exciting process is given by

$$\lambda(t, \mathcal{F}_t) = \exp(\omega) + \sum_{i\geq 1} 1_{[0,t]}(t_i) w(t - t_i),$$

where ω is a constant and w is some non-negative weighting function.

- Hawkes introduced the general class

$$\lambda(t, \mathcal{F}_t) = \mu(t) + \sum_{j=1}^{P} \sum_{i=1}^{\tilde{N}(t)} \alpha_j \exp(-\beta_j(t - t_i))$$

where $\alpha_j \geq 1$, $\beta_j \geq 1$, and $\mu > 0$ is a deterministic function. Here P is an integer

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$$\mathbb{E}[\lambda(t_i)|\mathcal{F}_{t_{i-1}}] = \mu(t_{t_{i-1}}) + \sum_{j=1}^{P} \sum_{k=1}^{i-1} \alpha_j \mathbb{E}[\exp(-\beta_j t_i)|\mathcal{F}_{t_{i-1}}] \exp(\beta_j t_k)$$

Point process

- The log likelihood can be computed on the basis of a recursion. In particular,

$$\log L(W,\theta) = \sum_{i=1}^{n}\left[\int_{t_{i-1}}^{t_i}\mu(s)ds - \sum_{j=1}^{P}\sum_{k=1}^{i-1}\frac{\alpha_j}{\beta_j}(1-\exp(-\beta_j(t_i-t_k)))\right] + \sum_{i+1}^{n}\left[\log\left(\mu(t_i)+\sum_{j=1}^{P}\alpha_j A_i^j\right)\right],$$

where

$$A_i^j = \sum_{k=1}^{i-1}\exp(-\beta_j(t_i-t_k)) = \exp(-\beta_j(t_i-t_{i-1}))(1+A_i^{j-1})$$

Outline

1. Order book dynamics
   - Order book as a point process
2. Market making problem(s)
   - Market making strategy
   - HJB equation
   - Solving HJB equation
3. References

Market maker

- Market makers supply optimal execution services for clients.
- Today HF market makers on many exchanges make up a substantial part of the total HFT activity.
- Market making and optimal portfolio liquidation are based on probabilistic models defined on certain reference probability spaces
- The increase in computer power has made it possible for HF market makers to deploy ever more complicated trading strategies to profit from changes in market conditions.
- By definition, a characteristic of HF market makers is that the strategies are designed to hold close to no inventories over very short periods of time, from seconds to at most one day, to avoid exposure both to markets after the close and to avoid posting collateral overnight

Market making strategy

- HF market makers profit from posting limit orders on both sides of the order book turning positions over very quickly to make a very small margin per round trip transaction.
- For HF market makers price anticipation and prediction concerning the order flow are important drivers of profit
- To devise an optimal schedule of a large order, or optimal portfolio liquidation, participants may choose a mixture of market and limit orders.
- The purpose of devising an optimal schedule of a large order, buy or sell, is to control the trading cost by balancing a trade-off between market impact and market risk (market impact demands trading to be slow while the presence of market risk favors faster trading).
- An optimal schedule of a large order may involve the use of market orders and limit orders in combination, as well as a routing of the orders to different exchanges

Market making strategy

- We consider a market maker who is only posting limit orders and who is allowed to control the ask and bid quotes, denoted by $p^+ = \{p_t^+\}_{t\in[0,T]}$ and $p^- = \{p_t^-\}_{t\in[0,T]}$, by continuously posting limit orders on both sides of the book
- The distance from the midprice is determined by the $\mathcal{F}_t$-adapted controls $\delta_t^+ = p_t^+ - S_t$ and $\delta_t^- = S_t - p_t^-$.
- Let $Q = \{Q_t\}_{t\in[0,T]}$ denote the inventory and X be the cash process
- We assume that the dynamics of Q_t and X_t are governed by

$$\begin{aligned} dQ_t &= dN_t^- - dN_t^+, \\ dX_t &= [S_t + \delta_t^+]dN_t^+ - [S_t - \delta_t^-]dN_t^-, \end{aligned}$$

  where $N^\pm = \{N_t^\pm\}_{t\in[0,T]}$ two independent Poisson processes with intensities $\lambda^\pm(\delta^\pm)$, which are nonincreasing functions determining the fill rates
- The wealth or profit and losses (PNL) of the market maker, at t, is then given by

$$PNL_t = X_t + Q_t S_t$$

Model

- Assume the mide-price $\{S_t\}_{t\in[0,T]}$ is descibed by

$$dS_t(\omega) = \mu_t(\omega)dt + \sigma_t(\omega)dW_t(\omega), \ \ t \in [0,T], \ \ \omega \in \Omega, \quad (1)$$

  where $W = \{W_t\}_{t\in[0,T]}$ is a Brownian motion defined on a $(\Omega, \mathcal{F}, P)$ with filtration $\mathbb{F} = \{\mathcal{F}_t\}$. μ and σ are $\{\mathcal{F}_t\}$-adapted.
- To study the situation where market participants consider the model defined by (1) as uncertain, model risk or uncertainty is incorporated into the model by assuming that for almost all $\omega \in \Omega$

$$-\bar{\mu} \leq \mu_t(\omega) \leq \bar{\mu}, \ \ \underline{\sigma} \leq \sigma_t(\omega) \leq \bar{\sigma}, \ \ \forall t \in [0,T], \quad (2)$$

  where $0 < \bar{\mu} < \infty$ and $0 < \underline{\sigma} \leq \bar{\sigma} < \infty$.
- Market participant captures model risk or uncertainty by using solely $0 < \bar{\mu}$, $\underline{\sigma}$ and $\bar{\sigma}$ in connection with (1), to make decisions.

Model

- Given a utility function $\psi(s,x,q)$, the value function associated with the market making problem under model risk or uncertainty is

$$v(t,s,x,q) = \sup_{\delta^+,\delta^-} \inf_{(\mu,\sigma)\in\mathcal{U}} E_{t,s,x,q}[\psi(S_T,X_T,Q_T)],$$

  where $\mathcal{U}$ denotes the set of al $\{\mathcal{F}_t\}$-adapted (μ,σ) satisfying (2).
- We assume that

$$\psi(s,x,q) = -\exp(-\gamma(x+qs-\eta|q|),$$

  where $\gamma,\eta \geq 0$, with γ measuring risk aversion and η representing a penalty on the inventory remaining at T.
- Also, the fill rates are assumed to be given by

$$\lambda^{\pm} = Ae^{-\rho\delta^{\pm}}, \quad \text{where } A>0, \text{ and } \rho>0,$$

HJB

- Let

$$\mathcal{L} = \mathcal{L}^{\mu,\sigma} = \mu_t \partial_s + \frac{1}{2}(\sigma_t)^2 \partial^2_{ss}$$

denote the operator associated with the model. The Hamilton-Jacobi-Bellman equation associated with the market making problem becomes (cf. Öksendahl & Sulem 2007)

$$\begin{aligned} 0 \quad = \quad & \sup_{\delta^+,\delta^-} \inf_{(\mu,\sigma)\in\mathcal{U}} \Big[\partial_t v(t,s,x,q) + \mathcal{L}^{\mu,\sigma} v(t,s,x,q) \\ & + \lambda^+(\delta^+)[v(t,s,x+(s+\delta^+),q-1) - v(t,s,x,q)] \\ & + \lambda^-(\delta^-)[v(t,s,x-(s-\delta^-),q+1) - v(t,s,x,q)]\Big] \end{aligned} \tag{3}$$

for $(t,s) \in (0,T) \times \mathbb{R}$ and $(x,q) \in \mathbb{R} \times \{-Q,\dots,Q\}$, with the terminal condition

$$v(T,s,x,q) = \psi(s,x,q)$$

HJB

- The problem (3) is easily seen to decouple into two independent optimization problems: first note that for fixed δ^- and δ^+, the step of initially taking the infimum with respect to $(\mu,\sigma) \in \mathcal{U}$ is equivalent to solving the optimization problem

$$\inf_{(\mu,\sigma)\in\mathcal{U}} \mathcal{L}^{\mu,\sigma} = \inf_{(\mu,\sigma)\in\mathcal{U}} \left[\mu_t \partial_s v + \frac{1}{2}(\sigma_t)^2 \partial^2_{ss} v \right],$$

which results in two separate problems,

$$\inf_{\mu\in[-\bar{\mu},\bar{\mu}]} [\mu_t \partial_s v], \quad \inf_{\sigma\in[\underline{\sigma},\bar{\sigma}]} [(\sigma_t)^2 \partial^2_{ss} v] \tag{4}$$

- The optimal controls is (4) are

$$\mu^*(t,s,x,q) = -\bar{\mu}\mathrm{sgn}(\partial_s v(t,s,x,q)), \quad \sigma^* = F(\partial^2_{ss} v),$$

where $F(x) = \underline{\sigma} 1_{z>0} + \bar{\sigma} 1_{z\leq 0}$.

- Then

$$\mathcal{L}^{\mu^*,\sigma^*} v(.) = H^*(t, \partial_s v(.), \partial^2_{ss} v(.)), \text{ where } H^*(t,p,r) = -\bar{\mu}|p| + \frac{1}{2}(F(r))^2$$

HJB

- The Hamilton-Jacobi-Bellman equation becomes

$$\begin{aligned} 0 &= \partial_t v(t,s,x,q) + H^*(t, \partial_s v(t,s,x,q), \partial^2_{ss} v(t,s,x,q)) \\ &+ \sup_{\delta^+} Ae^{-\rho\delta^+}[v(t,s,x+(s+\delta^+),q-1) - v(t,s,x,q)] \\ &+ \sup_{\delta^-} Ae^{-\rho\delta^-}[v(t,s,x-(s-\delta^-),q+1) - v(t,s,x,q)] \end{aligned} \quad (5)$$

Lemma

There exists a solution to the associated Hamilton?Jacobi?Bellman equation in (3)

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HJB: Solution

- The value function is given by

$$\begin{aligned} U(t,s,x,q) &= -\exp(-\gamma(x+qs-\eta|q|))u_q(t)^{-\frac{\gamma}{\rho}} \\ u(t) &= (u_{-Q},\dots,u_Q)(t) = e^{-E(T-t)}\bar{1}_{2Q+1}, \end{aligned} \tag{6}$$

  where E is a $(2Q+1)\times(2Q+1)$-dimensional matrix with 0 zero element exept for $E_{q,q-1},\ E_{q,q}, E_{q,q+1}$. $\bar{1}_{2Q+1}$ denotes the $(2Q+1)$-dimensional vector having all entries identical to 1.

- $\mu^*, sigma^*, \delta_*^+, \delta_*^-$ are given by

$$\begin{aligned} \mu^* &= -\bar{\mu}\,\mathrm{sgn}(q), \quad \sigma^* = \bar{\sigma} \\ \delta_*^+(t) &= \frac{1}{\gamma}\log(1+\frac{\gamma}{\rho}) + \frac{1}{\rho}\log\frac{u_q(t)}{u_{q-1}(t)} - \eta(|q|-|q-1|),\ q \neq -Q, \\ \delta_*^-(t) &= \frac{1}{\gamma}\log(1+\frac{\gamma}{\rho}) + \frac{1}{\rho}\log\frac{u_q(t)}{u_{q+1}(t)} - \eta(|q|-|q+1|),\ q \neq -Q \end{aligned}$$

Outline

References

- N. Hautsch Econometrics of Financial High-Frequency Data, Springer 2012
- R., Cont, and A. De Larrard, Order book dynamics in liquid markets: limit theorems and diffusion approximations (2011)
- K. Nyström and S. M. Ould Aly, A framework for the modeling of order book dynamics based on event sizes (2013)
-
- K. Nyström, S. M. Ould Aly and C: Zhang, Martket Making and Portfolio Liquidation under Uncertainty (2014)
- Á. Cartea, S. Jaimungal and J. Ricci Buy low sell high: A high frequency trading perspective (2014)
-
- F. Almgren and N. Chriss Optimal execution of portfolio transactions (2000)

Thank you!