Black-Scholes and Game Theory

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Established in collaboration with MIT

Sequential game

Two players: **Nature** and **Investor**

- Nature acts as an adversary, reveals state of the world S_t
- Investor acts by action a_t
- Investor incurs loss $l(a_t, S_t)$

Aim is to minimize regret, or rather perform well with respect to the best action in hindsight.

$$Regret = \sum_{t=1}^{n} l(a_t, S_t) - \inf_{a \in \mathcal{A}} \sum_{t=1}^{n} l(a, S_t)$$

Regret Learning - Minimax

Our interest is in minimax regret

$$Regret = \min_{a_1} \max_{S_1} \cdots \min_{a_n} \max_{S_n} \sum_{t=1}^{n} l(a_t, S_t) - \inf_{a \in \mathcal{A}} \sum_{t=1}^{n} l(a, S_t)$$

Probabilistic: $S_1, \cdots, S_n$ are iid
Worst-case: $S_1, \cdots, S_n$ are chosen adversarially

Regret defined above can be bounded above sub-linearly Blackwell[4]. Standard assumptions in regret learning lead to simple algorithms and upper bounds. Generally loss functions are convex - analysis is easier.

Black-Scholes and extensions

- Black-Scholes-Merton is now well developed technology
- At the heart is **perfect replication** and continous trading
- Perfect replication is a myth and reality is discrete
- Challenge is to produce features of market using game theory

Black-Scholes

We can trade the underlying stock and bond to replicate an option expiring at time T.
The self-financing replicating portfolio V_t equals the Black-Scholes price C_t

$$C_t = \mathbb{E}[g(S_T)|\mathcal{F}_t] = V_t, \quad \forall t \in [0, T]$$

These methods are well understood. Can the world of on-linearning offer something new? We can treat the above case for a generic *convex* payoff function $\mathbf{g}(\cdot)$.

Regret and Options

Imagine we are deciding between a replicating strategy and buying the option.

- We replicate the option by trading Δ_i amounts of underlying
- At the end of our strategy we compare the payoff of the derivative contract had we bought it and not replicated
- The difference between the two above is our regret for **not** buying the option

Abernethy et al. [1] develop an interesting approach where the the Black-Scholes value is seen as a game between **nature** and the **investor**

- Nature sets the price fluctuation r_i for each round: $S \mapsto S(1 + r_i)$
- Investor hedges the final payout by trading the underlying security $\$\Delta_i$ amount and receives $\Delta_i r_i$
- There are n rounds
- After nth round Investor is charged $g(S) = g(S_0 \cdot \prod_{i=1}^{n}(1 + r_i))$
- Variance Budget for $\sum_{i=1}^{n} r_i \leq c$ is c which is decreased round by round

Minimax[1]

An online hedging strategy is an algorithm that selects sequence of share purchares $\Delta_i' s$ (where $\Delta \in \mathbb{R}$) with the goal of minimizing

$$g(S_0 \cdot \prod_{i=1}^{n}(1 + r_i)) - \sum_{i=1}^{n} \Delta_i r_i \equiv \text{Hedging regret}$$

Hedging Regret is the difference between option value and the hedging strategy.

Minimax

We have n trading rounds and m rounds remaining: $n \in \mathbb{N}$ and $0 \leq m \leq n$. Total Variance budget c and jump constraint ζ per round.

$$V_\zeta^{(n)}(S;c;m) = \inf_{\Delta \in \mathbb{R}} \sup_r \{V_\zeta^{(n)}(S(1+r);c-r^2;m-1) - \Delta r\}$$

Base case is $V_\zeta^{(n)}(S;c;0) = g(S)$. We can think of V as the minimax price.

Minimax converges to BS

Under some technical conditions

$$\lim_{n \to \infty} \text{Minimax price}_n \to \text{Black-Scholes}$$

Hedging Regret is the difference between option value and hedging strategy.

Volatility games

Our aim

Introduce multiplayer zero-sum vol games.

Work in progress.

Volatility games

$game(K,T)$ corresponds to an option with strike K and maturity T.
Different strikes and maturities correspond to different points on the implied volatility surface. In our setting, c can be thought of as $\sigma_{imp}^2 \cdot (T-t)$, the total variance budget available at start of hedging.

Volatility games

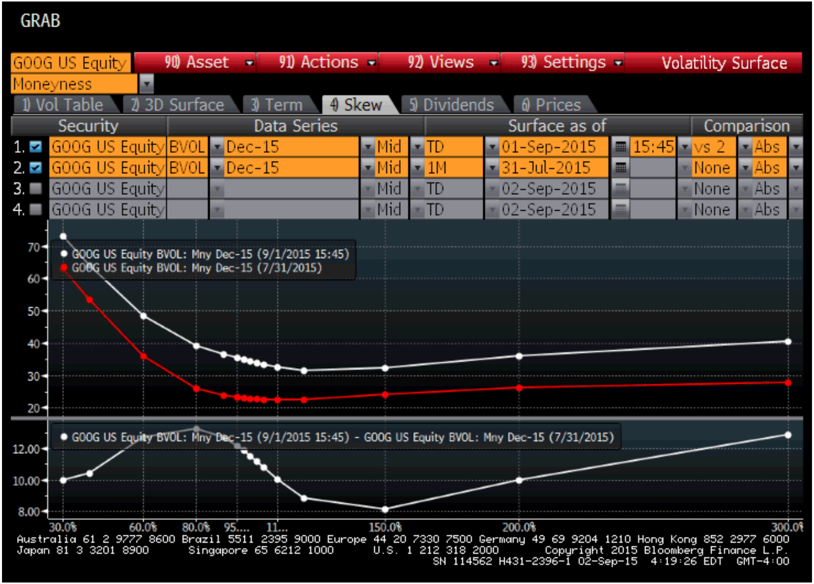
Imagine two different strikes K_1 and K_2 where $K_1 \neq K_2$. Then the variance budgets of these games is different. We assume a no-arbitrage vol surface.

Connection

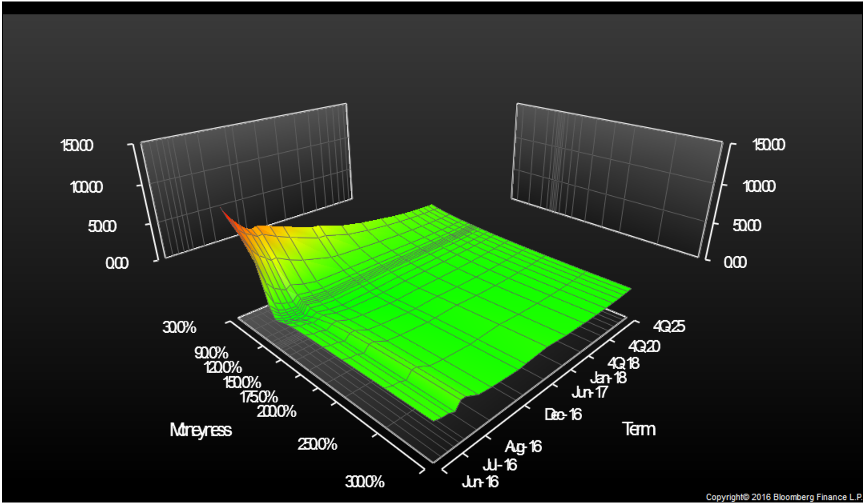
Diffferent strike games are consistent with vol surface.

What is the interpretation in game theory?

Volatility smirk



Alphabet Inc. Vol Surface



Implied Volatility games

Implied volatility surface $\leftrightsquigarrow game(K, T) \leftrightsquigarrow$ Variance Budget.

Static no-arbitrage conditions for the implied volatility model are well understood, some rough bounds can be obtained Hodges[2].

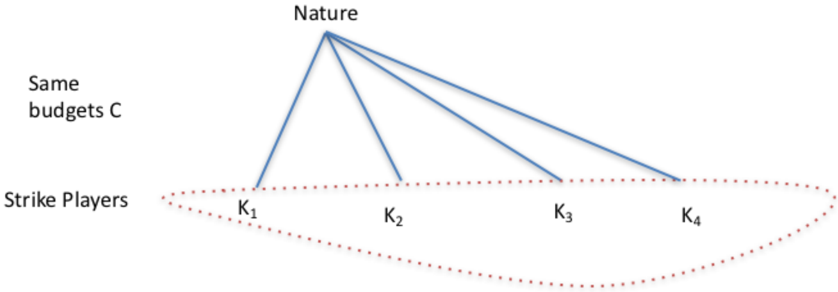
Implied Volatility games

How about dynamic no-arbitrage conditions and how they translate to different games. This is a challenging issue under traditional SDE models. Can game theory offer us something new.

We are aiming to extend the minmimax games to be consistent across different strikes and maturities.

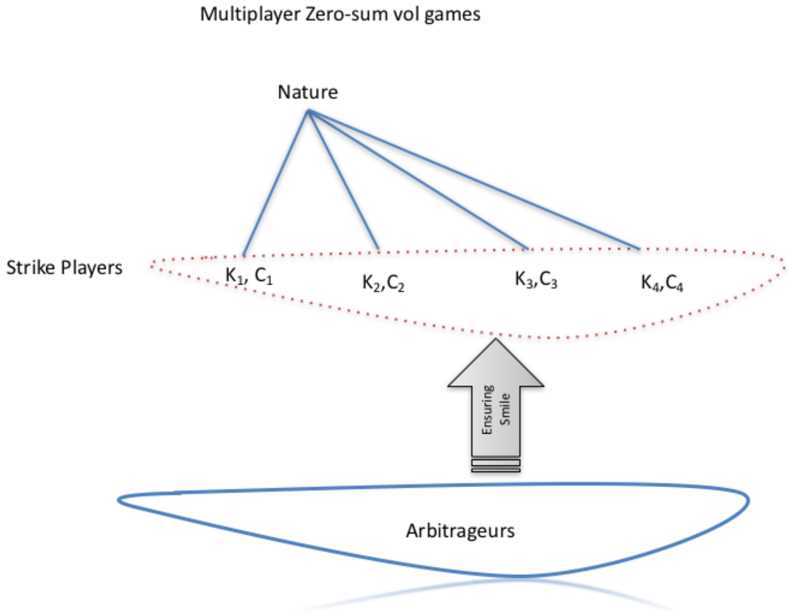
No Smile/Smirk

Multiplayer Zero-sum games



No smile

Smile/Smirk



Multiplayer Zero-sum vol games
Nature
Strike Players
K1, C1
K2,C2
K3,C3
K4,C4
Ensuring Smile
Arbitrageurs

Multi-player vol games

Protocol with assumption that the vol surface is given by market. We focus on a given time slice and think about the game.

- Nature vs Players K_1, K_2, K_3, K_4
- Each player plays a zero-sum game with Nature with variance budget c varying with strike
- Arbitrageurs exist who enforce the static no-arbitrage conditions of the vol surface

Smile games

If players deviate from vol smile variance budget then they open themselves up to infinite losses and the arbitrageurs can make infinite amounts of free money by locking into correcting trades.

Conclusion

- Extend to dynamic Volatility games
- Link game theoretic ideas to classical math finance
- Rich seam of techniques in probability and math finance which can be translated into game-theoretic setting
- Make the connection with game theory is a fruitful endeavor in its own right
- Zero sum vol surface connection still has technical conditions to be worked out

Thank you

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