Fake Brownian motion and calibration of a Regime Switching Local Volatility model

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Plan

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Fake Brownian motion

- A *fake* Brownian motion $(X_t)_{t\geq 0}$ is a continuous martingale that has the same marginal distributions as the Brownian motion $(W_t)_{t\geq 0}$ but is not a Brownian motion.
- Albin (2007) and Oleszkiewicz (2008) : explicit constructions of fake Brownian motions.
- Hobson (2009) : fake martingale diffusions.
- Stochastic processes matching given marginals is a question arising in mathematical finance.

Trying to match marginals

- The market gives the prices of European Calls $C(T_i, K_i)$ for some $T_i, K_i \geq 0$.

- A model $(S_t)_{t\geq 0}$ is calibrated to European options if

$$\forall T, K \geq 0, \;\; C(T,K) = \mathbb{E}\left[D_T\left(S_T - K\right)^+\right].$$

- By Breeden and Litzenberger (1978), {prices of European Call options for all $T, K > 0$} $\Longleftrightarrow$ {marginal distributions of $(S_t)_{t\geq 0}$}.

The Dupire Model

- Dupire Local Volatility model (1992), matching market marginals:

$$dS_t = rS_t dt + \sigma_{Dup}(t, S_t) S_t dW_t$$

$$\sigma_{Dup}(T, K) = \sqrt{2 \frac{\partial_T C(T, K) + rK\partial_K C(T, K)}{K^2 \partial^2_{KK} C(T, K)}}$$

- Modelization of volatility risk?
- Real market prices only on a finite set of (T_i, K_i) : robustness to interpolation?

LSV models

- Motivation: get processes with richer dynamics (e.g. take into account volatility risk) and satisfying marginal constraints.
- Alexander and Nogueira (2004) and Piterbarg (2006): Local and Stochastic Volatility (LSV) model

$$dS_t = rS_t + f(Y_t)\sigma(t, S_t)S_t dW_t$$

- "Adding uncertainty" to LV models by a random multiplicative factor $f(Y_t)$, $(Y_t)_{t\geq 0}$ is a stochastic process.

Calibration of LSV Models

- By Gyongy's theorem (1988), the LSV model is calibrated to $C(T, K), \forall T, K > 0$ if

$$\mathbb{E}\left[f^2(Y_t)|S_t\right]\sigma^2(t, S_t) = \sigma^2_{Dup}(t, S_t)$$

$$\sigma(t, x) = \frac{\sigma_{Dup}(t, x)}{\sqrt{\mathbb{E}\left[f^2(Y_t)|S_t = x\right]}}$$

- The obtained SDE is nonlinear in the sense of McKean:

$$dS_t = rS_t dt + \frac{f(Y_t)}{\sqrt{\mathbb{E}[f^2(Y_t)|S_t]}}\sigma_{Dup}(t, S_t)S_t dW_t.$$

Simulation results

- Madan and Qian, Ren (2007): solve numerically the associated Fokker-Planck PDE, and get the joint-law of (S_t, Y_t).
- Guyon and Henry-Labordère (2011): efficient calibration procedure based on kernel approximation of the conditional expectation.
- However, calibration errors seem to appear when the range of $f(Y)$ is too large.

Theoretical results

- Abergel and Tachet (2010): local in time existence using small perturbations on a compact.

- Global existence and uniquess to LSV models remain on open problem.

A simpler SDE

- Let Y be a r.v. with values in $\mathcal{Y} := \{y_1, ..., y_d\}$.
- We assume $\forall i \in \{1, ..., d\}, \; \alpha_i = \mathbb{P}(Y = y_i) > 0$.
- We study the SDE (FBM), with $f > 0$:

$$
\begin{aligned}
dX_t &= \frac{f(Y)}{\sqrt{\mathbb{E}\left[f^2(Y)|X_t\right]}} dW_t \\
X_0 &\sim \mu.
\end{aligned}
$$

- $X_0, Y, (W_t)_{t\geq 0}$ are independent.

The Fokker Planck system

- We define for $i \in \{1, ..., d\}$, $\lambda_i := f^2(y_i)$, $\lambda_{min} := \min_i \lambda_i$, $\lambda_{max} := \max_i \lambda_i$.

- For $i \in \{1, ..., d\}$, define p_i s.t., for $\phi \geq 0$ and measurable, $\mathbb{E}\left[\phi\left(X_t\right) 1_{\{Y=y_i\}}\right] = \int_{\mathbb{R}} \phi(x) p_i(t,x) dx$.

- The associated Fokker-Planck system is:

$$\forall i \in \{1, ..., d\}, \partial_t p_i = \frac{1}{2}\partial^2_{xx}\left(\frac{\sum_j p_j}{\sum_j \lambda_j p_j}\lambda_i p_i\right)$$
$$p_i(0) = \alpha_i \mu$$

- $\sum_j p_j$ is solution to the heat equation.

Existence to SDE (FBM) and fake Brownian motion

Theorem

Under Condition (C):

$$(C)\ :\ \sum_i \left(\frac{\lambda_i}{\lambda_{max}} + \frac{\lambda_{max}}{\lambda_i}\right) \vee \sum_i \left(\frac{\lambda_i}{\lambda_{min}} + \frac{\lambda_{min}}{\lambda_i}\right) < 2d + 4.$$

there exists a weak solution to the SDE (FBM).

Theorem

If f is not constant on $\mathcal{Y}$, then any solution to the SDE (FBM) is a fake Brownian motion.

Rewriting into divergence form

The system can be rewritten in divergence form:

$$\begin{pmatrix} \partial_t p_1 \\ \vdots \\ \partial_t p_d \end{pmatrix} = \frac{1}{2}\partial_x \left((I_d + M) \begin{pmatrix} \partial_x p_1 \\ \vdots \\ \partial_x p_d \end{pmatrix} \right).$$

$$\begin{aligned} M_{ii} &= \frac{\sum_{j \neq i} \lambda_j p_j \sum_j (\lambda_i - \lambda_l) p_l}{\left(\sum_j \lambda_j p_j\right)^2}, \\ M_{ik} &= \frac{\lambda_i p_i \sum_j (\lambda_j - \lambda_k) p_j}{\left(\sum_j \lambda_j p_j\right)^2}, \quad i \neq k. \end{aligned}$$

Computing standard energy estimates (S.E.E)

- Multiply the system by $(p_1, \dots, p_d)$, and integrate in x:

$$\frac{1}{2}\partial_t\left(\int_{\mathbb{R}}\sum_{i=1}^{d} p_i^2\,dx\right) = -\frac{1}{2}\int_{\mathbb{R}}(\partial_x p_1, \dots, \partial_x p_d)\,(I_d + M)\begin{pmatrix}\partial_x p_1\\ \cdot\\ \cdot\\ \partial_x p_d\end{pmatrix}dx.$$

- Goal : S.E.E. in $L^2([0,T], H^1(\mathbb{R})) \cap L^\infty([0,T], L^2(\mathbb{R}))$.

- We want (coercivity property):

$$\exists \epsilon > 0 \; s.t. \; \forall y \in \mathbb{R}^d, \;\; y^* M y \geq (\epsilon - 1)\,|y|^2.$$

M as a convex combination

- $\overline{\lambda} := \frac{\sum_j \lambda_j p_j}{\sum_j p_j}$, $w_j := \frac{\lambda_j p_j}{\sum_k \lambda_k p_k}$.
- $M_{ii} = \sum_{j \neq i} w_j \left(1 - \frac{\lambda_i}{\overline{\lambda}}\right)$, and if $i \neq k$, $M_{ik} = \sum_{j \neq i} w_j \left(1 - \frac{\lambda_i}{\overline{\lambda}}\right)$.
- Then $M = \sum_{j=1}^{d} w_j M_j$, where

$$M_j := \begin{pmatrix} \left(\frac{\lambda_1}{\overline{\lambda}} - 1\right) & & & & & & \\ & \ddots & & & & & \\ & & \left(\frac{\lambda_{j-1}}{\overline{\lambda}} - 1\right) & & & & \\ \left(1 - \frac{\lambda_1}{\overline{\lambda}}\right) & \cdots & \left(1 - \frac{\lambda_{j-1}}{\overline{\lambda}}\right) & 0 & \left(1 - \frac{\lambda_{j+1}}{\overline{\lambda}}\right) & \cdots & \left(1 - \frac{\lambda_d}{\overline{\lambda}}\right) \\ & & & & \left(\frac{\lambda_{j+1}}{\overline{\lambda}} - 1\right) & & \\ & & & & & \ddots & \\ & & & & & & \left(\frac{\lambda_d}{\overline{\lambda}} - 1\right) \end{pmatrix} \leftarrow \text{row } j.$$

How to have $\forall y \in \mathbb{R}^d, \; y^* M y \geq -|y|^2$

- Sufficient to study $M_j, \forall j, \forall \overline{\lambda} \in [\lambda_{min}, \lambda_{max}]$
- $a_i := \left(\frac{\lambda_i}{\overline{\lambda}} - 1\right) > -1$
- $y^* M_j y = \sum_{i \neq j} a_i \left(y_i^2 - y_i y_j\right)$
- Young's inequality : $-a_i y_i y_j \geq -(1 + a_i) y_i^2 - \frac{a_i^2}{4(1+a_i)} y_j^2$
- $y^* M_j y \geq -\left(\sum_{i \neq j} y_i^2\right) - \left(\sum_{i \neq j} \frac{\left(\lambda_i - \overline{\lambda}\right)^2}{4 \lambda_i \overline{\lambda}}\right) y_j^2$
- Sufficient condition:

$$\max_j \max_{\overline{\lambda} \in [\lambda_{min}, \lambda_{max}]} \left(\sum_{i \neq j} \frac{\left(\lambda_i - \overline{\lambda}\right)^2}{4 \lambda_i \overline{\lambda}} \right) \leq 1.$$

How to have $\forall y \in \mathbb{R}^d,\ y^* M y \geq -|y|^2$

- Equivalent formulation:

$$\max_j \max_{\overline{\lambda} \in [\lambda_{min}, \lambda_{max}]} \sum_{i \neq j} \left(\frac{\lambda_i}{\overline{\lambda}} + \frac{\overline{\lambda}}{\lambda_i} \right) \leq 2d + 2.$$

- Convexity of $\overline{\lambda} \to \frac{\lambda_i}{\overline{\lambda}} + \frac{\overline{\lambda}}{\lambda_i}$ on $[\lambda_{min}, \lambda_{max}]$:

$$\max_j \sum_{i \neq j} \left(\frac{\lambda_i}{\lambda_{min}} + \frac{\lambda_{min}}{\lambda_i} \right) \vee \max_j \sum_{i \neq j} \left(\frac{\lambda_i}{\lambda_{max}} + \frac{\lambda_{max}}{\lambda_i} \right) \leq 2d + 2.$$

- Sufficient condition:

$$\sum_{i} \left(\frac{\lambda_i}{\lambda_{min}} + \frac{\lambda_{min}}{\lambda_i} \right) \vee \sum_{i} \left(\frac{\lambda_i}{\lambda_{max}} + \frac{\lambda_{max}}{\lambda_i} \right) \leq 2d + 4.$$

Getting coercivity of M

- Remember coercivity property:

$$\exists \epsilon > 0 \; s.t. \; \forall y \in \mathbb{R}^d, \; y^* M y \geq (\epsilon - 1) \, |y|^2.$$

- Obtained if

$$(C) \; : \; \sum_i \left(\frac{\lambda_i}{\lambda_{max}} + \frac{\lambda_{max}}{\lambda_i} \right) \vee \sum_i \left(\frac{\lambda_i}{\lambda_{min}} + \frac{\lambda_{min}}{\lambda_i} \right) < 2d + 4.$$

- Ensures that the range $f^2(Y)$ is not too large.

Fact

M satisfies the coercivity property if and only if (C) holds.

Step 1/3: Existence to an approximate PDS when $\mu \in L^2(\mathbb{R})$

- Assume that $\mu(dx) = p_0(x)dx$, $p_0 \in L^2(\mathbb{R})$.
- For $\epsilon > 0$, use Galerkin's method to solve an approximate PDE:

$$\begin{aligned}
\begin{pmatrix} \partial_t p_1^\epsilon \\ \cdot \\ \cdot \\ \partial_t p_d^\epsilon \end{pmatrix} &= \frac{1}{2}\partial_x \left((I_d + M^\epsilon) \begin{pmatrix} \partial_x p_1^\epsilon \\ \cdot \\ \cdot \\ \partial_x p_d^\epsilon \end{pmatrix} \right) \\
(p_1^\epsilon(0), \ldots, p_d^\epsilon(0)) &= (\alpha_1, \ldots, \alpha_d)\, p_0
\end{aligned}$$

Step 1/3: Existence to an approximate PDS when $\mu \in L^2(\mathbb{R})$

$$M_{ii}^{\epsilon} = \frac{\sum_{j \neq i} \lambda_j \left(p_j^{\epsilon}\right)^{+} \sum_j (\lambda_i - \lambda_l) \left(p_l^{\epsilon}\right)^{+}}{\left(\epsilon \vee \sum_j \lambda_j \left(p_j^{\epsilon}\right)^{+}\right)^2},$$

$$M_{ik}^{\epsilon} = \frac{\lambda_i \left(p_i^{\epsilon}\right)^{+} \sum_j (\lambda_j - \lambda_k) \left(p_j^{\epsilon}\right)^{+}}{\left(\epsilon \vee \sum_j \lambda_j \left(p_j^{\epsilon}\right)^{+}\right)^2}, \quad i \neq k.$$

- Taking p_{ϵ}^{-} as test function, we show that $p_{\epsilon} \geq 0$.
- $\forall \epsilon, \forall i, \sum_j M_{ji}^{\epsilon} = 0 \implies \sum_j p_j^{\epsilon} > 0$.
- $\epsilon \to 0$, existence of a solution to the original PDS.

Step 2/3: Existence to the PDS when $\mu \in \mathcal{P}(\mathbb{R})$

- By mollification of μ, we use the results of Step 1 to extract a solution to the PDS when $\mu \in \mathcal{P}(\mathbb{R})$.
- We use the fact that $\sum_j p_j$ is solution to the heat equation.

Step 3/3: Existence to the SDE for $\mu_0 \in \mathcal{P}(\mathbb{R})$

We use the results of Figalli (2008):

$$\exists \text{Fokker-Planck solution } (\mu_t)_{t\geq 0}$$

$$\implies \exists \text{martingale solution with marginals given by } (\mu_t)_{t\geq 0},$$

to prove existence to

$$\begin{aligned} dX_t &= \frac{f(Y)}{\sqrt{\mathbb{E}\left[f^2(Y)|X_t\right]}} dW_t \\ X_0 &\sim \mu \end{aligned}$$

with $X_0, Y, (W_t)_{t\geq 0}$ independent.

Presentation

- We consider the following dynamics (RSLV):

$$dS_t = rS_t dt + \frac{f(Y_t)}{\sqrt{\mathbb{E}\left[f^2(Y_t)|S_t\right]}}\sigma_{Dup}(t, S_t)S_t dW_t,$$

where $(Y_t)_{t\geq 0}$ takes values in $\mathcal{Y}$, and

$$\mathbb{P}\left(Y_{t+dt} = y_j | Y_t = y_i, \log S_t = x\right) = q_{ij}(x)dt.$$

- Switching diffusion, special case of LSV model.
- Jump distributions and intensities are functions of the asset level.

Assumptions

- (C), (Coerc. 1): f satisfies condition (C).
- (HQ), (Bounded I) $\exists \overline{q} > 0,\ s.t.\ \forall x \in \mathbb{R}, |q_{ij}(x)| \leq \overline{q}$.

We define $\tilde{\sigma}_{Dup}(t,x) := \sigma_{Dup}(t, e^x)$.

- (H1), (Bounded vol.) $\tilde{\sigma}_{Dup} \in L^\infty([0,T], W^{1,\infty}(\mathbb{R}))$.
- (H2), (Coerc. 2) $\exists \underline{\sigma} > 0$ s.t. $\underline{\sigma} \leq \tilde{\sigma}_{Dup}$ a.e. on $[0,T] \times \mathbb{R}$,.
- (H3), (Regul. 1) $\exists \eta \in (0,1], \exists H_0 > 0$, s.t. $\forall s,t \in [0,T], \forall x,y \in \mathbb{R}$,

$$|\tilde{\sigma}_{Dup}(s,x) - \tilde{\sigma}_{Dup}(t,y)| \leq H_0 \left(|x-y|^\eta + |t-s|^\eta\right).$$

- (H4), (Regul. 2) for a.e. $x \in \mathbb{R}$,

$$\partial_x \sigma_{Dup}(s,x) \underset{s \to t}{\rightarrow} \partial_x \sigma_{Dup}(t,x)$$

Main results

Theorem

Under Conditions (H1)-(H4), (HQ) and (C) there exists a weak solution to the SDE (RSLV).

- The proof is an adaptation of the proof for SDE (FBM) combined with an extension of the results of Figalli for a jumping process.

Summary

- We obtained existence of a class of fake Brownian motions by
  1. Solving the associated Fokker-Planck PDE,
  2. Linking with existence of martingale solution and SDEs.

- With similar arguments, we obtained existence of calibrated RSLV models :

$$dS_t = rS_t dt + \frac{f(Y_t)}{\sqrt{\mathbb{E}\left[f^2(Y_t)|S_t\right]}}\sigma_{Dup}(t, S_t)S_t dW_t,$$

where for Y, jump intensities and laws depend on X_t.

Thank you!

Thank you for your attention!