Optimal multiple stopping problems; application in pricing of multi-exercise options

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July 1, 2016

Outline

- Monte Carlo method
- Multilevel Monte Carlo method
- Swing Options
- Optimal stopping problems
- valuation of swing options (work in progress)

Monte Carlo Method

Using Monte Carlo method for estimating $\mathbb{E}[P]$:

$$Y = \frac{1}{N} \sum_{i=1}^{N} P(\omega^{(i)}),$$

where ωs are N independent samples and Y is an unbiased estimator of $\mathbb{E}[P]$.

According to the central limit theorem when $N \to \infty$, the error becomes normally distributed with variance $\frac{1}{N}\mathbb{V}[P]$.

Monte Carlo Method

While $\hat{Y}$ and $\hat{P}$ be approximations of Y and P respectively, the Mean Square Error is:

$$\mathbb{E}[(\hat{Y} - \mathbb{E}[P])^2] = \frac{1}{N}\mathbb{V}[\hat{P}] + (\mathbb{E}[\hat{P}] - \mathbb{E}[P])^2.$$

For more accuracy, we need to increase N and make smaller $\mathbb{E}[\hat{P}] - \mathbb{E}[P]$.

Multilevel Monte Carlo Method

Consider P_1, P_2, ..., P_L are approximations of P_L. Instead of Using Monte Carlo Method for estimating $\mathbb{E}[P_L]$ we use:

$$\mathbb{E}[P_L] = \mathbb{E}[P_0] + \sum_{l=1}^{L} (\mathbb{E}[P_l] - \mathbb{E}[P_{l-1}]).$$

Therefore an unbiased estimator for $\mathbb{E}[P_L]$ by using Monte Carlo estimator:

$$\frac{1}{N_0} \sum_{n=1}^{N_0} P_0^{(0,n)} + \sum_{l=1}^{L} \frac{1}{N_l} \sum_{n=1}^{N_l} (P_l^{(l,n)} - P_{l-1}^{(l,n)}), \tag{1}$$

where L is the number of higher level, level l in (l, n) indicates that independent samples are used at each level of correction.

Multilevel Monte Carlo Method

Estimating $\mathbb{E}[P_l] - \mathbb{E}[P_{l-1}]$ via Monte Carlo method in Eq.(1) in a common probability space,

- making strong correlation between them,
- leading to small variance for all the higher levels,
- needing few samples to estimate.

Multilevel Monte Carlo Method

Theorem

Let P denote a random variable, and let P_l denote the corresponding level l numerical approximation. If there exist independent estimators Y_l based on N_l Monte Carlo samples, and positive constants $\alpha, \beta, \gamma, c_1, c_2, c_3$ such that $\alpha \geq \frac{1}{2} min(\beta, \gamma)$ and

1- $\mathbb{E}[P_l - P] \leq c_1 2^{-\alpha l}$,

2-

$$\mathbb{E}[Y_l] = \begin{cases} \mathbb{E}[P_0], & l = 0, \\ \mathbb{E}[P_l - P_{l-1}], & l \geq 0, \end{cases}$$

3- $Var[Y_l] \leq c_2 N_l^{(-1)} 2^{-\beta l}$,

4- $\mathbb{E}[c_l] \leq c_3 N_l 2^{\gamma l}$, where c_l is the computational complexity of Y_l. Then there exists a positive constant c_4, such that for any $\epsilon < e^{-1}$ there are values L and N_l for which the multilevel estimator

Multilevel Monte Carlo Method

Theorem(continue)

$$Y = \sum_{l=0}^{L} Y_l,$$

has a mean square error with bound

$$\mathbb{E}[(Y - \mathbb{E}[P])^2] < \epsilon^2$$

, with a computational complexity C with bound:

$$\mathbb{E}[C] \leq \begin{cases} c_4 \epsilon^{-2}, & \beta > 1, \\ c_4 \epsilon^{-2} (log \epsilon)^2, & \beta = 1, \\ c_4 \epsilon^{-2-(1-\beta)/\alpha}, & 0 < \beta < 1 \end{cases} .$$

Swing Options

swing options are a class of multiple early exercise options that are traded in energy market, particularly in the natural gas and electricity markets.

These contracts provide option holders multiple exercise rights, and the holders can choose to exercise the rights on several given exercise dates before the end of the contract. However, on each exercise date, the holder can only exercise at most once. Furthermore, there could be penalties if the exercise volume exceeds some given constraint. There are several different versions of swing options, depending on the payoff features, the penalty constraints, the flexibility of exercise volumes, etc.

Swing Option Contracts

- Contract can be exercised at times $0 = t_1 < t_2 < \ldots < t_n = T$.
- δ is refracting time, time period between two exercise times.
- N is number of rights, $N \leq n$.
- K is the strike price.
- ξ_{t_i} is the volume of exercise at time t_i with constraint $\xi_{t_i} \in [p_i, q_i]$, and total volume constraint$\sum_{i=1}^{N} \xi_{t_i} \in [P, Q]$.
- $\phi_i(\xi_{t_i})$ is the penalty function at time t_i, and the total penalty is $\Phi(\sum_{i=1}^{N} \xi_{t_i})$ when the total volume .

Swing Option's Payoff

Let X_t be the price process follows a risk neutral model.
Value of swing option:

$$Max\mathbb{E}[\sum_{i=1}^{N} e^{-rt_i}(\xi_{t_{d_i}} G(S_{t_{d_i}}) - \phi_{t_{d_i}}(\xi_{t_{d_i}}))] - \Phi(\sum_{i=1}^{N} \xi_{t_i}),$$

Where $G(S_{t_{d_i}})$ is the payoff function, and $[d_1, \dots, d_N] \subseteq [1, 2, \dots, n]$.

If we let $N = n, K_t = K, p_i = p_0, q_i = q_0$ be constant, and assume that the holder is not allowed to exercise volumes out of constraints, and consider a call swing option, we will have the value of option:

$$Max\mathbb{E}[\sum_{i=1}^{N} e^{-rt_i}\xi_{t_i}(S_{t_i} - K)_+].$$

Optimal Stopping Problem

The theory of optimal stopping problem is considered with the problem of choosing a time to take a given action based on sequentially observed random variables in order to maximize an expected payoff or minimize an expected cost.

Let

- $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space,
- $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$ be a filtration,
- $X = \{X_t\}_{t \geq 0}$ be a non-negative $\mathbb{F}$-adapted process:
  X is continuous,
  $\mathbb{E}[\bar{X}] < \infty$ where $\bar{X} := sup_{t \geq 0} X_t$.
- $\mathcal{S}_T$ be the collection of all stopping times with values in $[0, T]$,
- given $\theta \in \mathcal{S}_T$,

$$\mathcal{S}_{\theta,T} = \{\tau; \tau \geq \theta \quad a.s.\}$$

.

Optimal Stopping Problem

Classical optimal stopping problem:

$$\hat{X}_0 = sup_{\tau \in \mathcal{S}_T} \mathbb{E}[X_\tau]$$

.

Defining inductively the sequence of Snell envelope, according to solve optimal stopping problem:

$$Y_t^{(0)} = 0,$$

$$Y_t^{(i)} = esssup_{\tau \in \bar{\mathcal{S}}_t} \mathbb{E}[X_\tau^{(i)} | \mathcal{F}_t], \quad \forall i = 1, 2, \ldots, N,$$

where

$$\bar{\mathcal{S}}_t = \{\tau \in \mathcal{S}_T ; \tau \geq t\}.$$

Optimal Stopping Problem

The ith exercise reward process $X^{(i)}$ is given by:

$$X_t^{(i)} = X_t + \mathbb{E}[Y_{t+\delta}^{(i-1)}|\mathcal{F}_t], \qquad for 0 \leq t \leq T - \delta,$$

and

$$X_t^{(i)} = X_t, \qquad for T - \delta < t \leq T.$$

Identifying a set of optimal stopping times for the multiple stopping problem:
Set

$$\tau_1^* = inf\{t \geq 0; Y_t^{(N)} = X_t^{(N)}\}.$$

Define

$$\tau_i^* = inf\{t \geq \delta + \tau_{i-1}^*; Y_t^{(N-i+1)} = X_t^{(N-i+1)}\} 1_{\delta+\tau_{i-1}^* \leq T}, \quad for \quad 2 \leq i \leq N.$$

Optimal Stopping Problem

It is proved that $X_0^{(N)}$ can be calculated by solving inductively N single optimal stopping problems sequentially.

$$\vec{\tau}^* = (\tau_1^*, \tau_2^*, \dots, \tau_N^*) \in \mathcal{S}_0^{(N)}.$$

Theorem

If we assume that:

- the filtration $\mathbb{F}$ is left continuous,
- $\mathbb{E}[\bar{X}^p] < \infty \quad for \quad some \quad p > 1$,
- the process X is cont. a.s.,
- $\mathbb{E}[\bar{X}] < \infty$ where $\bar{X} := sup_{t \geq 0} X_t$,

  then

$$Z_0 = Y_0^P = \mathbb{E}[X_{\vec{\tau}^*}].$$

Valuing Swing Options

For valuation of swing options, following optimal multiple stopping problem is introduced:

$$Z_0 = X_0^{(N)} := sup_{(\tau_1,\tau_2,\ldots,\tau_n)\in\mathcal{S}_0^{(N)}} \mathbb{E}[\sum_{i=1}^{N} X_{\tau_i}].$$

For valuing swing options in finite horizon and discrete time case, for example consider maturity time is 1 year, value function of the multiple stopping problem is:

$$\nu_N(0, X_0) := sup_{\vec{\tau}\in\bar{\mathcal{S}}^{(N)}} \sum_{i=1}^{N} \mathbb{E}[e^{-r\tau_i}\Phi(X_{\tau_i})],$$

assume that $\Phi(x) = (K - x)_+$.

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Thank you