On Conditional Quantiles Approximation for Elliptical Random Fields

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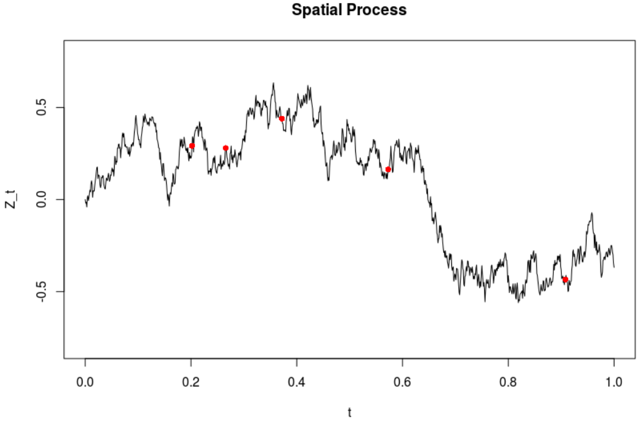
Context

Let $(Z_t)_{t \in T}$ a random field observed in n points $t_1, \ldots, t_n$.



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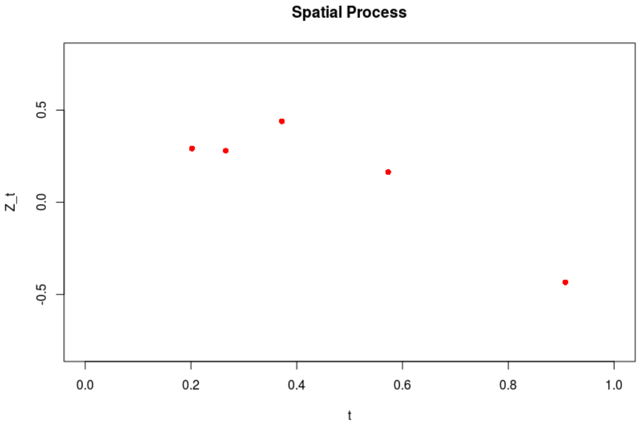


Context

Kriging (Krige, 1952 [5], Matheron, 1963 [6]) estimates the conditional mean of the process at point $x \in T$, using :

$$\hat{\mathbb{E}}[Z_x|Z] = {\beta^*}^T Z$$

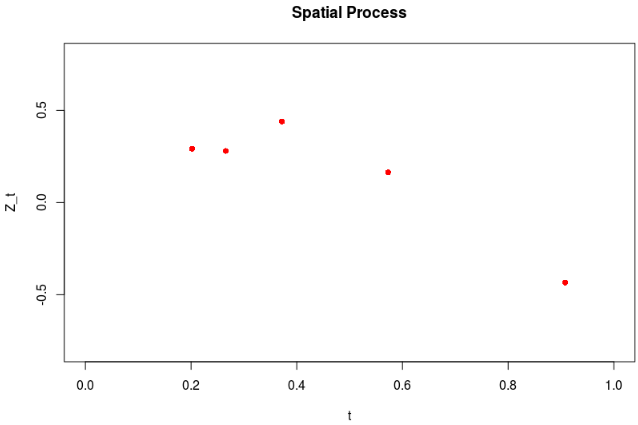
where $Z = (Z_{t_1}, ..., Z_{t_n})$, and β^* is the solution of :

$$\arg\min_{\beta \in \mathbb{R}^n} \mathbb{E}[(Z_x - \beta^T Z)^2]$$

i.e $\beta^* = \mathbb{E}[Z^T Z]^{-1}\mathbb{E}[ZZ_x]$.

Context

Back to our example.



Context

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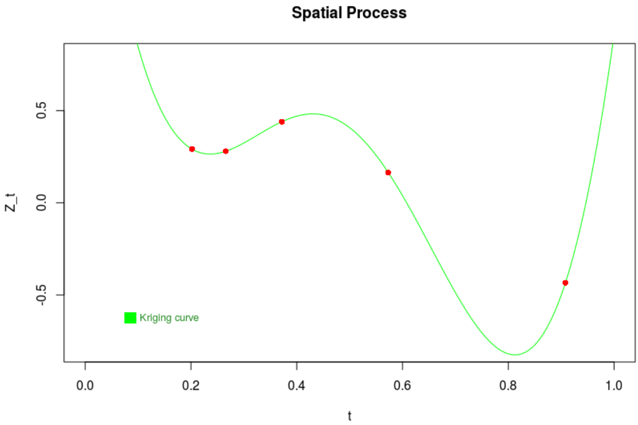
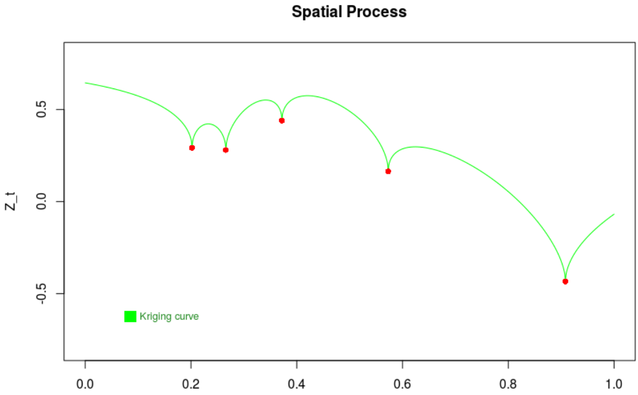


Context

Question : Can we estimate conditional quantiles $q_\alpha(Z_x|Z)$, for non necessarily gaussian random fields ?



Elliptical Distribution

Definition

Let X a $d-$dimensional random vector. X is elliptical if and only if there exists a unique $\mu \in \mathbb{R}^d$, a semi-positive definite matrix $\Sigma \in \mathbb{R}^{d\times d}$, and a function $\Phi : \mathbb{R}_+ \to \mathbb{R}$ such that the characteristic function of $X - \mu$ can be written
$\mathbb{E}\left[e^{it(X-\mu)}\right] = \Phi(t'\Sigma t)$.

Definition

X is said consistent (or has the consistency property) if Φ is unrelated to d.

Representation theorem

Theorem (Cambanis, Huang, Simons, 1981 [1])

$X \sim \mathcal{E}_d(\mu, \Sigma)$ if and only if :

$$X \stackrel{d}{=} \mu + R\Lambda U^{(d)}$$

with $\Lambda\Lambda^T = \Sigma$, $U^{(d)}$ is a $d-$dimensional random vector uniformly distributed on $\mathcal{S}^{d-1}$ (the unit disk of dimension d), R is a non-negative random variable being stochastically independent of $U^{(d)}$.

- R is called the radius of X.

Consistent elliptical radius

Theorem (Kano, 1994 [3])

$X \sim \mathcal{E}_d(\mu, \Sigma, R)$ has the consistency property if and only if :

$$R \stackrel{d}{=} \frac{\chi_d}{\epsilon}$$

where ϵ is a positive random variable unrelated to d and χ_d.

Elliptical Distribution

Theorem (Elliptical density)

If $X \sim \mathcal{E}_d(\mu, \Sigma, R)$, then X has the following form of density :

$$f_X(x) = \frac{c_d}{|\det(\Sigma)|^{\frac{1}{2}}} g_d\left((x-\mu)\Sigma^{-1}(x-\mu)\right)$$

where $g_d(t) = \frac{\Gamma\left(\frac{d}{2}\right)}{2\pi^{\frac{d}{2}}} \sqrt{t}^{-(d-1)} f_R(\sqrt{t})$, and $f_R(t)$ is the p.d.f of R.

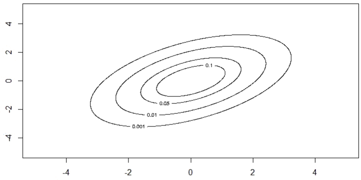
- g_d is called the generator of X.

Examples

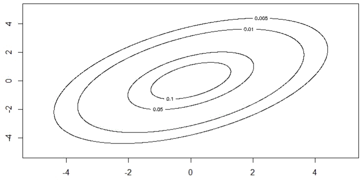
Distribution	Coefficient c_d	Generator $g_d(t)$	Radius R
Gaussian	$\frac{1}{(2\pi)^{\frac{d}{2}}}$	$\exp(-\frac{t}{2})$	χ_d
Student, $\nu > 0$	$\frac{\Gamma(\frac{d+\nu}{2})}{\Gamma(\frac{\nu}{2})} \frac{1}{(\nu\pi)^{\frac{d}{2}}}$	$\left(1 + \frac{t}{\nu}\right)^{-\frac{d+\nu}{2}}$	$\frac{\sqrt{\nu}\chi_d}{\chi_\nu}$
Gaussian Mixture	$\frac{1}{(2\pi)^{\frac{d}{2}}}$	$\sum_{k=1}^{n} \pi_k \theta_k^d \exp\left(-\frac{\theta_k^2}{2} t\right)$	$\frac{\chi_d}{\sum_{k=1}^{n} \pi_k \theta_k}$
Uniform Mixture	$\frac{\Gamma\left(\frac{d+1}{2}\right)}{\sqrt{2}\pi^{\frac{d}{2}}}$	$\frac{\chi^2_{d+1}(t)}{t^{\frac{d+1}{2}}}$	$\frac{\chi_d}{\mathcal{U}(]0,1[)}$

Examples

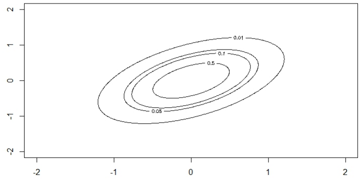
Bivariate Gaussian Distribution



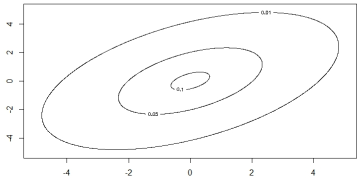
Bivariate t-Student Distribution



Bivariate Unimodal Gaussian Mixture Distribution



Bivariate Uniform Mixture Distribution



Affine transformation

Proposition (Affine transformation)

Let X a consistent $(R,d)-$elliptical random vector with parameters μ and Σ. Then $\forall c \in \mathbb{R}^d, b \in \mathbb{R}$, $c^T X + b$ is $(R,1)-$elliptical with parameters $c^T\mu + b$ and $c^T\Sigma c$.

Subvectors distributions

Proposition (Subvectors distributions)

Let X a consistent $(R, d)-$elliptical random vector with parameters μ and Σ. We define X_1 and X_2 respectively d_1 and d_2-dimensional subvectors of X, such as $d_1 + d_2 = d$. Let us write Σ :

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$$

Then X_1 and X_2 are respectively $(R, d_1)-$ and $(R, d_2)-$elliptical with parameters μ_1, Σ_{11} and μ_2, Σ_{22}, respectively.

Conditional distributions

Proposition (Conditional distribution)

We consider the previous hypothesis. We can deduce the conditional parameters of $X_2|(X_1 = x_1)$ *:*

$$\begin{cases} \mu_{2|1} = & \mu_2 + \Sigma_{21}\Sigma_{11}^{-1}(x_1 - \mu_1) \\ \Sigma_{2|1} = & \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12} \end{cases}$$

Furthermore, $X_2|(X_1 = x_1)$ *is still elliptical, with radius* R^* *given by :*

$$R^* \stackrel{d}{=} R\sqrt{1-\beta}\,|\left(R\sqrt{\beta}U^{(d)} = C_{11}^{-1}(x_1 - \mu_1)\right)$$

where C_{11} *is the Cholesky root of* Σ_{11}*, and* $\beta \sim Beta(\frac{d_1}{2}, \frac{d_2}{2})$.

Conditional distributions

Proposition

At last, the conditional density of $X_2|(X_1 = x_1)$ is given by :

$$f_{X_2|X_1}(x_2|x_1) = \frac{c_{2|1}}{|\Sigma_{2|1}|^{\frac{1}{2}}} g_d\left(q_1 + (x_2 - \mu_{2|1})^T \Sigma_{2|1}^{-1}(x_2 - \mu_{2|1})\right)$$

with $c_{2|1} = \frac{c_d}{c_{d_1} g_{d_1}(q_1)}$, *and* $q_1 = (x_1 - \mu_1)^T \Sigma_{11}^{-1}(x_1 - \mu_1)$.

Conditional Quantiles

If we define :

$$\Phi_R(x) = \mathbb{P}\left(RU^{(1)} \leq x\right)$$

Proposition (Conditional elliptical quantile)

Let X a $(R, N+1)-$elliptical random vector with parameters μ and Σ. We define X_1 and X_2 respectively N and $1-$dimensional subvectors of X. Let us write Σ :

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$$

Then the $\alpha-$quantile of $X_2|(X_1 = x_1)$ is given by :

$$q_\alpha\left(X_2|X_1 = x_1\right) = \mu_{2|1} + \sqrt{\Sigma_{2|1}}\Phi_{R^*}^{-1}(\alpha)$$

Gaussian example

$$X_2|(X_1 = x_1) \sim \mathcal{N}(\mu_{2|1}, \Sigma_{2|1})$$

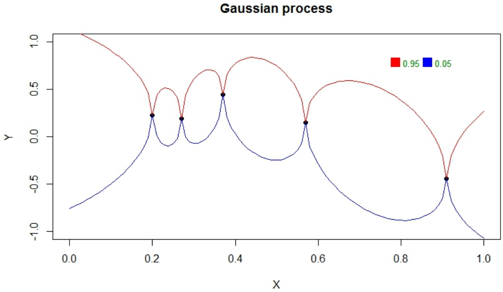


Figure: 0.95 and 0.05−quantiles for a conditional gaussian process : illustration in the dimension $d = 1$

Student example

Lemma

Let X a $d-$dimensional Student distribution with ν degrees of freedom, and parameters $\mu \in \mathbb{R}^d$ and $\Sigma \in \mathbb{R}^{d \times d}$. Let X_1 and X_2, respectively d_1 and d_2-dimensional subvectors of X, such as $d_1 + d_2 = d$. Then the density function of the conditional random variable $X_2|(X_1 = x_1)$ is :

$$f_{X_2|X_1}(x_2|x_1) = \frac{\Gamma\left(\frac{\nu+d}{2}\right)}{((\nu+d_1)\pi)^{\frac{d_2}{2}}\Gamma\left(\frac{\nu+d_1}{2}\right)|\Sigma_{2|1}|^{\frac{1}{2}}}\left[1+\frac{1}{\nu}\frac{q_{2|1}(x_2)}{1+\frac{1}{\nu}q_1}\right]^{-\frac{\nu+d}{2}}\left[\frac{\frac{\nu+d_1}{\nu}}{1+\frac{1}{\nu}q_1}\right]^{\frac{d_2}{2}}$$

where $q_{2|1}(x_2)$ and q_1 are the Mahalanobis distances :

$$\begin{cases} q_{2|1}(x_2) = & (x_2-\mu_{2|1})^T\Sigma_{2|1}^{-1}(x_2-\mu_{2|1}) \\ q_1 = & (x_1-\mu_1)^T\Sigma_{11}^{-1}(x_1-\mu_1) \end{cases}$$

Student example

Proposition (Conditional Student quantile)

The conditional $\alpha-$quantile of $X_2|(X_1 = x_1)$, $X_2 \in \mathbb{R}$, $X_1 \in \mathbb{R}^N$, is :

$$q_\alpha(X_2|X_1 = x_1) = \mu_{2|1} + \sqrt{\Sigma_{2|1}}\sqrt{\frac{\nu}{\nu+N}}\sqrt{1 + \frac{1}{\nu}q_1}\,\Phi^{-1}_{\nu+N}(\alpha)$$

Student example

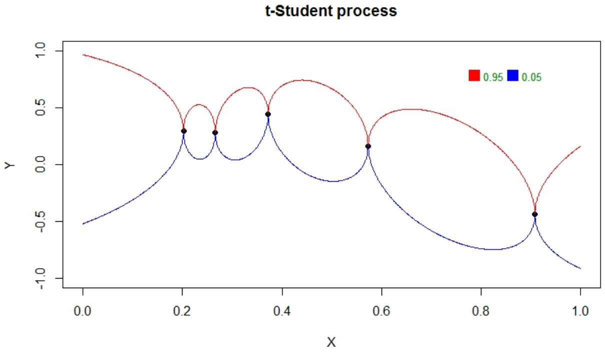


Figure: 0.95 and 0.05−quantiles for a conditional Student process with $\nu = 3$: illustration in the dimension $d = 1$

Quantile Regression [4]

$$\hat{q}_\alpha(X_2|X_1 = x_1) = {\beta^*}^T x_1 + \beta_0^*$$

where β^* and β_0^* are the solutions of the following minimization problem :

$$(\beta^*, \beta_0^*) = \underset{\beta\in\mathbb{R}^N, \beta_0\in\mathbb{R}}{\arg\min} \; \mathbb{E}[\phi_\alpha(X_2 - \beta^T X_1 - \beta_0)]$$

with the scoring function ϕ_α :

$$\phi_\alpha(x) = (\alpha - 1)x\mathbb{1}_{\{x<0\}} + \alpha x\mathbb{1}_{\{x>0\}}$$

Quantile Regression

Proposition (Optimal β^*)

The optimal β^* is given by :

$$\beta^* = \Sigma_{11}^{-1}\Sigma_{12}$$

Quantile Regression

Theorem (Quantile Regression Estimator)

The quantile regression vector (β^*, β_0^*) *of* $X_2|(X_1 = x_1)$ *is given by the following formulas :*

$$\begin{cases} \beta^* = & \Sigma_{11}^{-1}\Sigma_{12} \\ \beta_0^* = & \mu_2 - \Sigma_{21}\Sigma_{11}^{-1}\mu_1 + \sigma_{2|1}\Phi_R^{-1}(\alpha) \end{cases}$$

Then the Quantile Regression Estimator with level $\alpha \in [0, 1]$ *is given by :*

$$\hat{q}_\alpha(X_2|X_1 = x_1) = \mu_{2|1} + \sigma_{2|1}\Phi_R^{-1}(\alpha)$$

Furthermore,

$$\hat{q}_\alpha(X_2|X_1) \sim \mathcal{E}_1\left(\mu_2 + \sigma_{2|1}\Phi_R^{-1}(\alpha), \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}, R\right)$$

Gaussian example

$$\begin{cases} q_\alpha(X_2|X_1 = x_1) = & \mu_{2|1} + \sigma_{2|1}\Phi^{-1}(\alpha) \\ \hat{q}_\alpha(X_2|X_1 = x_1) = & \mu_{2|1} + \sigma_{2|1}\Phi^{-1}(\alpha) \end{cases}$$

The Quantile Regression Estimator (QRE) gives exactly the conditional quantiles.

Student example

$$\begin{cases} q_\alpha(X_2|X_1=x_1) = & \mu_{2|1} + \sigma_{2|1}\sqrt{\frac{\nu}{\nu+N}}\sqrt{1+\frac{1}{\nu}q_1}\Phi^{-1}_{\nu+N}(\alpha) \\ \hat{q}_\alpha(X_2|X_1=x_1) = & \mu_{2|1} + \sigma_{2|1}\Phi^{-1}_{\nu}(\alpha) \end{cases}$$

The error may be huge, especially if the Mahalanobis distance $q_1 = (x_1-\mu_1)^T\Sigma_{11}^{-1}(x_1-\mu_1)$ is tall.

Student example

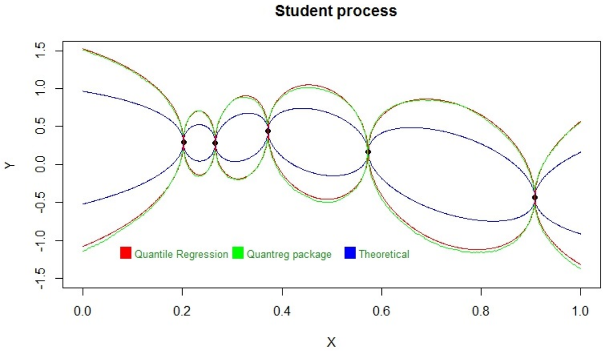
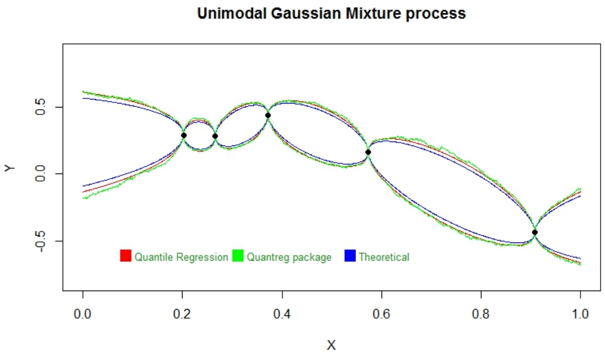


Figure: Theoretical quantiles in blue, quantiles get with the R package in green, and QRE in red, for levels $\alpha = 0.05$ and $\alpha = 0.95$

Gaussian Mixture example



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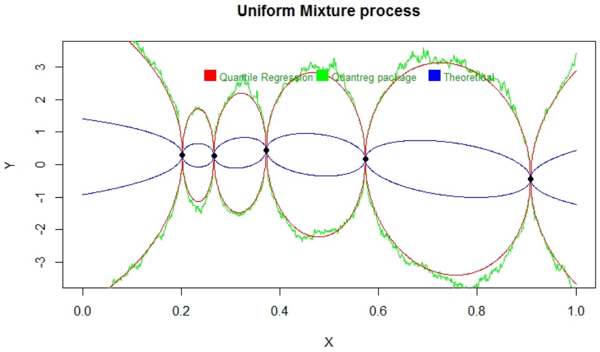
Uniform Mixture example



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Motivation

- Quantile Regression Estimator does not lead to conditional quantiles, in general
- Can we propose a correction, at least for extremal levels of α ?

$$\begin{cases} \hat{q}_\alpha(X_2|X_1 = x_1) & = \mu_{2|1} + \sigma_{2|1}\Phi_R^{-1}(\alpha) \\ q_\alpha(X_2|X_1 = x_1) & = \mu_{2|1} + \sigma_{2|1}\Phi_{R^*}^{-1}(\alpha) \end{cases}$$

- Can we find asymptotic equivalences between $\Phi_R^{-1}(\alpha)$ and $\Phi_{R^*}^{-1}(\alpha)$?

Assumption

Assumption

There exist $0 < \ell < +\infty$ *and* $\gamma \in \mathbb{R}$ *such as :*

$$\lim_{x \to +\infty} \frac{1-\Phi_{R^*}(x)}{1-\Phi_R(x^\gamma)} = \lim_{x \to +\infty} \frac{c_{N+1} g_{N+1}(q_1 + x^2)}{c_1 c_N g_N(q_1) \gamma x^{\gamma-1} g_1(x^{2\gamma})} = \ell$$

- Kind of equivalence between the distribution functions Φ_R and Φ_{R^*}
- We need an equivalence between Φ_R^{-1} and $\Phi_{R^*}^{-1}$ (Djurčić, Torgašev, 2001 [2])

Djurčić, Torgašev, 2001 [2]

Definition

A function f is a $\varphi-$function if $f : [0, +\infty[\to [0, +\infty[$, $f(0) = 0$, f is continuous, non decreasing on $[0, +\infty[$, and $f(x) \to +\infty$ when $x \to +\infty$.

Definition

K_c is the set of all $\varphi-$functions f with the property :

$$\lim_{\substack{x\to+\infty \\ \lambda\to 1}} \frac{f(\lambda x)}{f(x)} = 1$$

Djurčić, Torgašev, 2001 [2]

Lemma (Djurčić, Torgašev, 2001 [2])

Suppose that f and g are two strictly increasing $\varphi-$functions, at least one of the functions f^{-1}, g^{-1} belongs to the class K_c, and $f(x) \underset{x\to\infty}{\sim} g(x)$. Then $f^{-1}(x) \underset{x\to\infty}{\sim} g^{-1}(x)$

Extremal Quantile Estimators

Definition (Extreme Conditional Quantiles Estimators)

$$\begin{cases} \hat{\hat{q}}_{\alpha\uparrow}(X_2|X_1=x_1) = & \mu_{2|1} + \sigma_{2|1}\left[\Phi_R^{-1}\left(1 - \frac{1}{\frac{\ell}{1-\alpha}+2(1-\ell)}\right)\right]^{\frac{1}{\gamma}} \\ \hat{\hat{q}}_{\alpha\downarrow}(X_2|X_1=x_1) = & \mu_{2|1} - \sigma_{2|1}\left[\Phi_R^{-1}\left(1 - \frac{1}{\frac{\ell}{\alpha}+2(1-\ell)}\right)\right]^{\frac{1}{\gamma}} \end{cases}$$

Theorem (Equivalences of ECQE)

$$\begin{cases} q_\alpha(X_2|X_1=x_1) \underset{\alpha\to1}{\sim} \hat{\hat{q}}_{\alpha\uparrow}(X_2|X_1=x_1) \\ q_\alpha(X_2|X_1=x_1) \underset{\alpha\to0}{\sim} \hat{\hat{q}}_{\alpha\downarrow}(X_2|X_1=x_1) \end{cases}$$

Examples

Distribution	γ	ℓ
Gaussian	1	1
Student, $\nu > 0$	$\frac{N+\nu}{\nu}$	$\frac{\Gamma\left(\frac{\nu+N+1}{2}\right)\Gamma\left(\frac{\nu}{2}\right)}{\Gamma\left(\frac{\nu+N}{2}\right)\Gamma\left(\frac{\nu+1}{2}\right)}\left(1+\frac{q_1}{\nu}\right)^{\frac{N+\nu}{2}}\frac{\nu^{\frac{N}{2}+1}}{\nu+N}$
Gaussian Mixture	1	$\frac{\min(\theta_1,\ldots,\theta_n)^N \exp\left(-\frac{\min(\theta_1,\ldots,\theta_n)^2}{2}q_1\right)}{\sum_{k=1}^{n}\pi_k\theta_k^N\exp\left(-\frac{\theta_k^2}{2}q_1\right)}$
Uniform Mixture	$N+1$	$\frac{\Gamma\left(\frac{N+2}{2}\right)q_1^{\frac{N+1}{2}}\sqrt{2}}{\Gamma\left(\frac{N+1}{2}\right)(N+1)\chi^2_{N+1}(q_1)}$

Gaussian example

- $\gamma = \ell = 1$. Hence :

$$\begin{cases} \hat{\hat{q}}_{\alpha\uparrow}(X_2|X_1 = x_1) = & \mu_{2|1} + \sigma_{2|1}\Phi_R^{-1}(\alpha) \\ \hat{\hat{q}}_{\alpha\downarrow}(X_2|X_1 = x_1) = & \mu_{2|1} + \sigma_{2|1}\Phi_R^{-1}(\alpha) \end{cases}$$

- $\hat{\hat{q}}_{\alpha\uparrow}$ and $\hat{\hat{q}}_{\alpha\downarrow}$ are exactly the theoretical quantiles

Student example

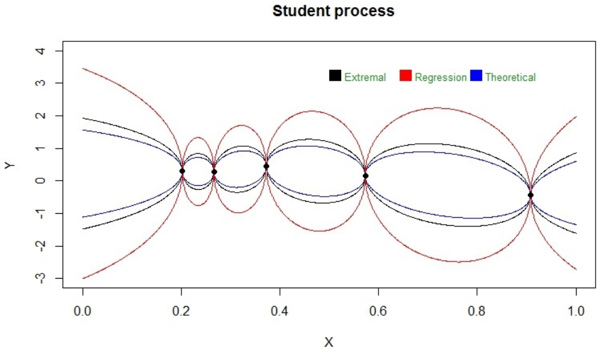


Figure: Theoretical quantiles in blue, extremal estimators in black, and QRE in red, for $\alpha = 0.995$ and 0.005

Gaussian Mixture example

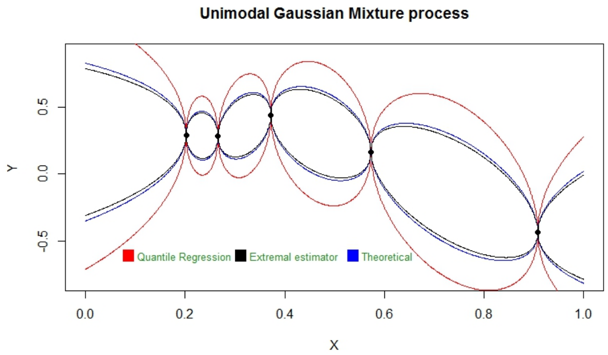


Figure: Theoretical quantiles in blue, extremal estimators in black, and QRE in red, for levels $\alpha = 0.005$ and $\alpha = 0.995$

Uniform Mixture example

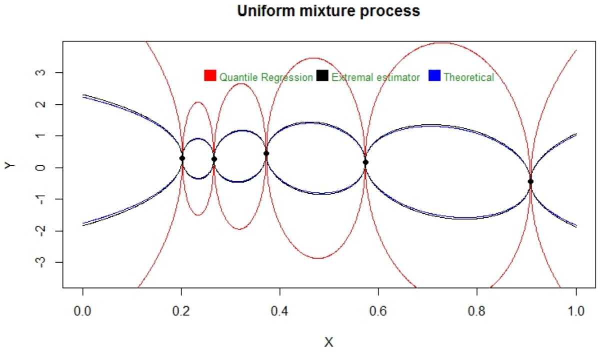


Figure: Theoretical quantiles in blue, extremal estimators in black, and QRE in red, for levels $\alpha = 0.005$ and $\alpha = 0.995$

Numerical study

We define the following *RMSE*s :

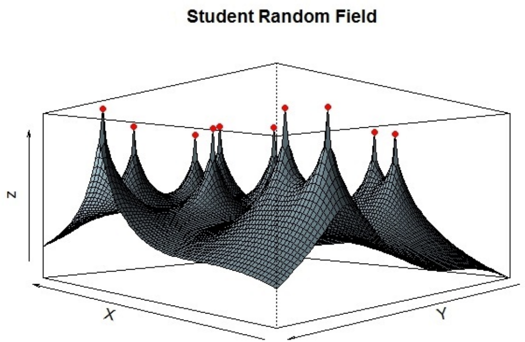
$$\begin{cases} RMSE(\hat{q}_\alpha) = & \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(q_\alpha(X_2^{(i)}|X_1 = x_1) - \hat{q}_\alpha(X_2^{(i)}|X_1 = x_1)\right)^2} \\ RMSE(\hat{\hat{q}}_\alpha) = & \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(q_\alpha(X_2^{(i)}|X_1 = x_1) - \hat{\hat{q}}_\alpha(X_2^{(i)}|X_1 = x_1)\right)^2} \end{cases}$$

Numerical study

α	Student		UGM		Uniform	
	$RMSE(\hat{q}_\alpha)$	$RMSE(\hat{\hat{q}}_\alpha)$	$RMSE(\hat{q}_\alpha)$	$RMSE(\hat{\hat{q}}_\alpha)$	$RMSE(\hat{q}_\alpha)$	$RMSE(\hat{\hat{q}}_\alpha)$
0.5	0	0	0	0	0	0
0.6	0.02717082	0.3716682	0.00146743	0.01868868	0.06691511	0.3335765
0.7	0.05902606	0.3758591	0.003233866	0.03562366	0.1511828	0.3008545
0.8	0.1044727	0.3583675	0.005926889	0.05054183	0.2968964	0.2494973
0.9	0.1952414	0.322185	0.01223556	0.06109096	0.7701059	0.1766719
0.95	0.3136118	0.2888097	0.0236387	0.0610452	1.880313	0.1234383
0.9995	2.880308	0.1478775	0.1628414	0.0003795359	250.1717	0.02022995
0.999995	16.54578	0.08062824	0.1093942	7.130243e-06	25178.53	0.00812057

Higher dimensions

- All the examples are in dimension $d = 1$, but the results are true in higher dimensions.



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Thank you for your attention !