Local volatility models enhanced with jumps

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Joint work with Pierre Henry-Labordère [1]

[1] http://papers.ssrn.com/sol3/papers.cfm?abstract_id=2781102

Mathematical analysis/algorithm

- Two models:
  - LVM with jumps:

$$\frac{dS_t}{S_t} = \sigma(t, S_t)\, dW_t + (J - 1)\,(dN_t - \lambda_t dt)$$

  - Regime-switching model:

$$\begin{aligned}\frac{dS_t}{S_t} &= \sigma_1(t, S_t)\, dW_t + (J - 1)\,(dN_t - \lambda_t dt)\,, \quad \forall t \leq \tau \\ \frac{dS_t}{S_t} &= \sigma_2(t, S_t)\, dW_t\,, \quad \forall t > \tau, \quad \sigma_2 := J_S \sigma_1\end{aligned}$$

- Leads to nonlinear McKean SDEs
- Monte-Carlo algorithm: particle method.
- Numerical results.

Calibration condition

- Consider the following dynamics:

$$\frac{dS_t}{S_t} = \sigma(t, S_t)\, dW_t + (J(S_t) - 1)\,(dN_t - \lambda_t dt)$$

- Formally: $S_t \sim \mathbb{P}_t^{\mathrm{mkt}}$ iff

$$\sigma^2(t, K) = \sigma^2_{\mathrm{Dupire}}(t, K) - 2\frac{\Lambda(t, K)}{K^2 \partial_{KK} C^{\mathrm{mkt}}(t, K)}.$$

where:

$$\Lambda(t, K) \equiv \mathbb{E}\left[\lambda_t (K - J(S_t) S_t)\left(1_{S_t > K} - 1_{J(S_t) S_t > K}\right)\right]$$

The case of deterministic intensity

In the case of a deterministic intensity $\lambda_t := \lambda(t)$, we have the following closed form for the local volatility:

$$\sigma^2(t,K) = \sigma^2_{\text{Dupire}}(t,K) - 2\lambda(t)\frac{\mathbb{E}^{\mathbb{P}^{\text{mkt}}}\left[(K - J(S_t)S_t)\left(1_{S_t>K} - 1_{J(S_t)S_t>K}\right)\right]}{K^2\partial_{KK}C^{\text{mkt}}(t,K)}.$$

$\sigma(t,K)$ is a function of market call prices.

McKean SDE

Theorem 1

Let us consider the following McKean SDE in $\mathbb{R}^n$

$$dX_t = b(X_{t-}, \mathbb{P}_t)dt + \sigma(X_{t-}, \mathbb{P}_t)dW_t + a(X_{t-}, \mathbb{P}_t)(dN_t - \lambda_t dt) \quad (1)$$

If the functions

$$\begin{aligned} \sigma, a &: \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n) \to \mathbb{R}^n \times \mathbb{R}^n, \\ b &: \mathbb{R}^n \times \mathcal{P}_2(\mathbb{R}^n) \to \mathbb{R}^n \end{aligned}$$

are Lipschitz-continuous:

$$|f(X, \mathbb{P}) - f(Y, \mathbb{Q})| \leq C|X - Y| + d(\mathbb{P}, \mathbb{Q}), \ \forall f \in \{b, \sigma, a\}$$

where d *is the Wasserstein distance, then the non linear SDE admits a unique solution* X_t *such that* $\mathbb{E}[\sup_{0 \leq t \leq T} |X_t|^2] < \infty$.

McKean SDE

Definition

Let $\mathcal{P}_2(\mathbb{R}^n)$ denote the set of probability measures on $\mathbb{R}^n$ with finite second order moments. For $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^n)$, the Wasserstein metric is defined by the formula,

$$d(\mu, \nu) = \inf_{(X,Y)} \{ \mathop{\mathbb{E}}_{\substack{X \sim \mu \\ Y \sim \nu}} [(X - Y)^2] \}$$

Example: The function

$$f(\mathbb{P}) = \mathbb{E}^{\mathbb{P}}[g(X)]$$

where g is Lipschitz-continuous for the canonical metric on $\mathbb{R}^n$, is Lipschitz-continuous for the Wasserstein distance.

McKean SDE: Regularizing the SDE

- Define the regularized volatility $\sigma^{\epsilon}(t, K)$ by:

$$\sigma^{\epsilon}(t,K)^2 \equiv \left(\sigma^2_{\text{Dupire}} - 2\frac{\mathbb{E}^{\mathbb{P}_t}[\lambda_t f^{\epsilon}(K,S_t)]}{K^2\partial^2_K C^{\text{mkt}}}\right) 1_{\left(\sigma^2_{\text{Dupire}} - 2\frac{\mathbb{E}^{\mathbb{P}_t}[\lambda_t f^{\epsilon}(K,S_t)]}{K^2\partial^2_K C^{\text{mkt}}}\right) > \eta} + \eta 1_{\left(\sigma^2_{\text{Dupire}} - 2\frac{\mathbb{E}^{\mathbb{P}_t}[\lambda_t f^{\epsilon}(K,S_t)]}{K^2\partial^2_K C^{\text{mkt}}}\right) \le \eta}$$

where $\lim_{\epsilon\to 0} f^{\epsilon}(K,S) = (K - J(S)S)(1_{S>K} - 1_{J(S)S>K})$, $\eta < (1-\alpha)/2$.

Proposition 2

Let us define

$$\frac{dS^{\epsilon}_t}{S^{\epsilon}_t} = \sigma^{\epsilon}\left(t, S^{\epsilon}_t\right) dW_t + \left(J(S^{\epsilon}_t) - 1\right)\left(dN_t - \lambda_t dt\right)$$

Then this non-linear SDE admits a unique solution S^{ϵ}_t.

Assumptions

- There exists $\alpha < 1$:

$$\lambda_{\max} \leq \alpha \frac{\partial_t C^{\mathrm{mkt}}(t,K)}{\mathbb{E}_t^{\mathbb{P}^{\mathrm{mkt}}}\left[(K - J(S_t)S_t)\left(1_{S_t>K} - 1_{J(S_t)S_t>K}\right)\right]}, \quad \forall\,(t,K)$$

- The function J is bounded from below by $J_{\min}$ and there exists $S_{\min} < S_{\max}$ such that:

$$\forall S \in [0, S_{\min}] \cup [S_{\max}, \infty) : \quad J(S) = 1.$$

Main theorem

Theorem 3

For all $(t, K) \in [0, T] \times \mathbb{R}_+$*:*

$$\lim_{\epsilon \to 0} \mathbb{E}[(S_t^\epsilon - K)^+] = C^{\mathrm{mkt}}(t, K).$$

Proof in two steps:
(i) $u_\epsilon(t, K) = \mathbb{E}[(S_t^\epsilon - K)^+] - C^{\mathrm{mkt}}(t, K)$ solves the right PDE family (E_ϵ)
(ii) (E_ϵ) admits only one solution that tends to 0 as $\epsilon \to 0$.

Proof: Step 1

Define (E_ϵ):

$$\begin{aligned}
\partial_t u_\epsilon(t,K) &= \frac{K^2}{2}\sigma^\epsilon(t,K)^2\partial_K^2 u_\epsilon + \left(\eta K^2 \partial_K^2 C^{\text{mkt}} - h(u_\epsilon)(t,K)\right)_+ + g_\epsilon(t,K)\\
u_\epsilon(0,K) &= 0, \quad \forall K \in \mathbb{R}_+
\end{aligned}$$

where

$$\begin{aligned}
g_\epsilon(t,K) &\equiv \mathbb{E}\left[\lambda_t\left((K - J(S_t^\epsilon)S_t^\epsilon)\left(1_{S_t^\epsilon > K} - 1_{J(S_t^\epsilon)S_t^\epsilon > K}\right) - f^\epsilon(K, S_t^\epsilon)\right)\right]\\
h(u_\epsilon)(t,K) &\equiv \partial_t C^{\text{mkt}} - \mathbb{E}^{\mathbb{P}_t^{\text{mkt}}}[\lambda_\epsilon(t,S_t)f^\epsilon(K,S_t)]\\
&- \int \lambda_\epsilon(t,s) f^\epsilon(K,s)\partial_s^2 u_\epsilon(t,s)ds,\\
\lambda_\epsilon(t,s) &\equiv \mathbb{E}[\lambda_t | S_t^\epsilon = s]
\end{aligned}$$

Proof: Step 2

Lemma 4

There exists a solution $u_\epsilon \in C^{1,2}$ (where u_ϵ and its first derivative $K\partial_K u_\epsilon$ cancel at infinity and 0) to PDE (2) for ϵ small enough and such that:

$$\lim_{\epsilon \to 0} u_\epsilon(t, K) = 0, \quad \forall (t, K) \in [0, T] \times \mathbb{R}_+.$$

This solution satisfies the PDE:

$$\partial_t v_\epsilon = \frac{K^2}{2} \sigma^\epsilon(t, K)^2 \partial_K^2 v_\epsilon + g_\epsilon(t, K), \quad v_\epsilon(0, K) = 0, \forall K.$$

Particle method: MC simulation

- Replace the law $\mathbb{P}_t$ by $\mathbb{P}_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{X_t^{i,N}}$, where the particles $(X_t^{i,N})_{1\leq i\leq N}$ are solutions of the $(\mathbb{R}^n)^N$-dimensional SDE:

$$
\begin{aligned}
dX_t^{i,N} &= b(t, X_{t-}^{i,N}, \mathbb{P}_t^N)dt + \sigma(t, X_{t-}^{i,N}, \mathbb{P}_t^N)dW_t^i \\
&+ a(t, X_{t-}^{i,N}, \mathbb{P}_t^N)(dN_t^i - \lambda_t^i dt), \quad X_0^{i,N} \in \mathbb{R}^n.
\end{aligned}
$$

- Prove that for all functions $\phi \in C_b(\mathbb{R}^n)$:

$$
\frac{1}{N}\sum_{i=1}^N \phi(X_t^{i,N}) \underset{N\to\infty}{\longrightarrow} \int_{\mathbb{R}^n} \phi(x)p(t,x)dx, \; a.s
$$

- We prove instead the L^1 convergence using the propagation of chaos.

Numerical computations

Consider N independent particles with coordinates $(S^i, \lambda^i, N^i)_{i=1,\ldots,N}$ following

$$\frac{dS_t^i}{S_t^i} = \sigma\left(t, S_t^i\right) dW_t^i + (J-1)\left(dN_t^i - \lambda_t^i dt\right)$$

Algorithm Divide the interval $[0, T]$ into intervals of size Δ.

1. $t := 0$, set $\sigma(t, S) = \sigma_{\text{Dupire}}(t, S)$ between 0 and Δ and diffuse the N particles up to Δ - say with an Euler discretization scheme.
2. Compute $\Lambda(\Delta, K)$ using Monte Carlo:

$$\Lambda(\Delta, K) = \frac{1}{N}\sum_{i=1}^{N} \lambda_\Delta^i (K - JS_\Delta^i)(1_{S_\Delta^i > K} - 1_{S_\Delta^i J(S_\Delta^i) > K})$$

   Then, compute the local volatility $\sigma(\Delta, S)$ for all S using Equation (1), and set $\sigma(t, S) = \sigma(\Delta, S)$ for all $t \in [\Delta, 2\Delta]$.
3. Iterate up to a maturity T.

Numerical implementation: $\rho = 0$

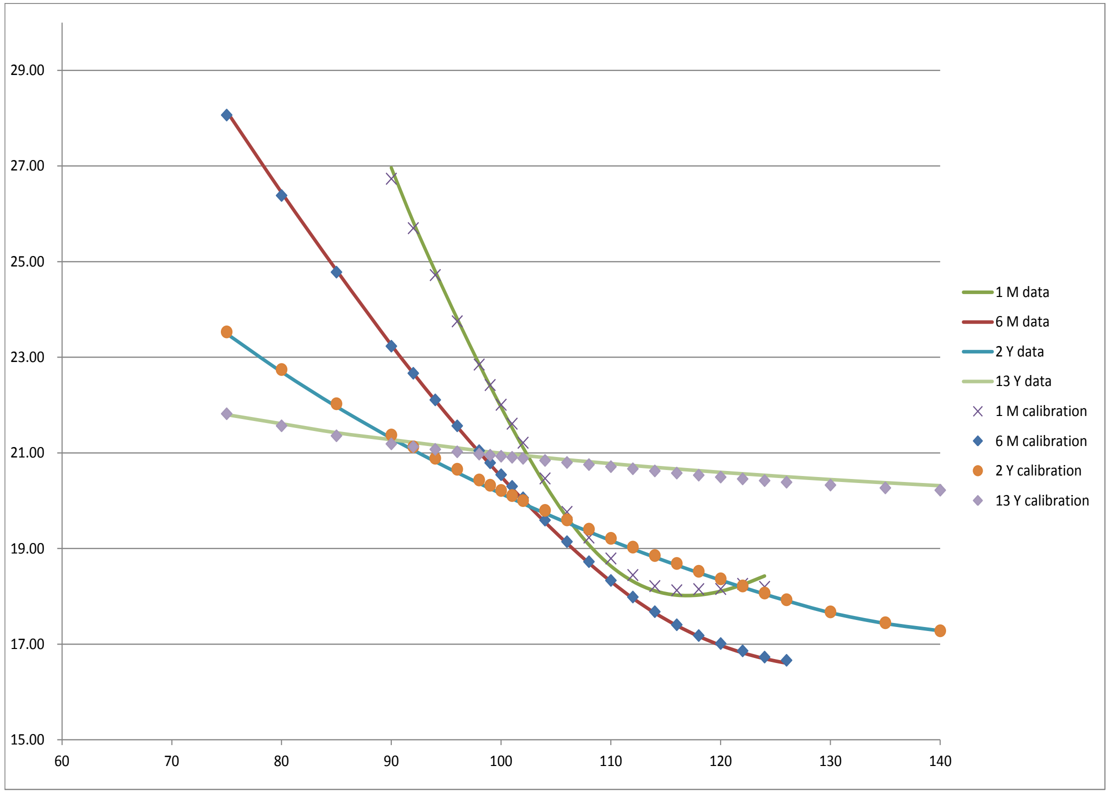

Figure: Calibrated implied volatilities compared to implied volatilities (DAX, 5-May-2015), $J = 0.9$, $\sigma = 100\%$, $\rho = 0$. We used 5000 particles to compute the local volatility, 40000 simulations to reprice vanillas.

Numerical implementation: $\rho = 0.4$

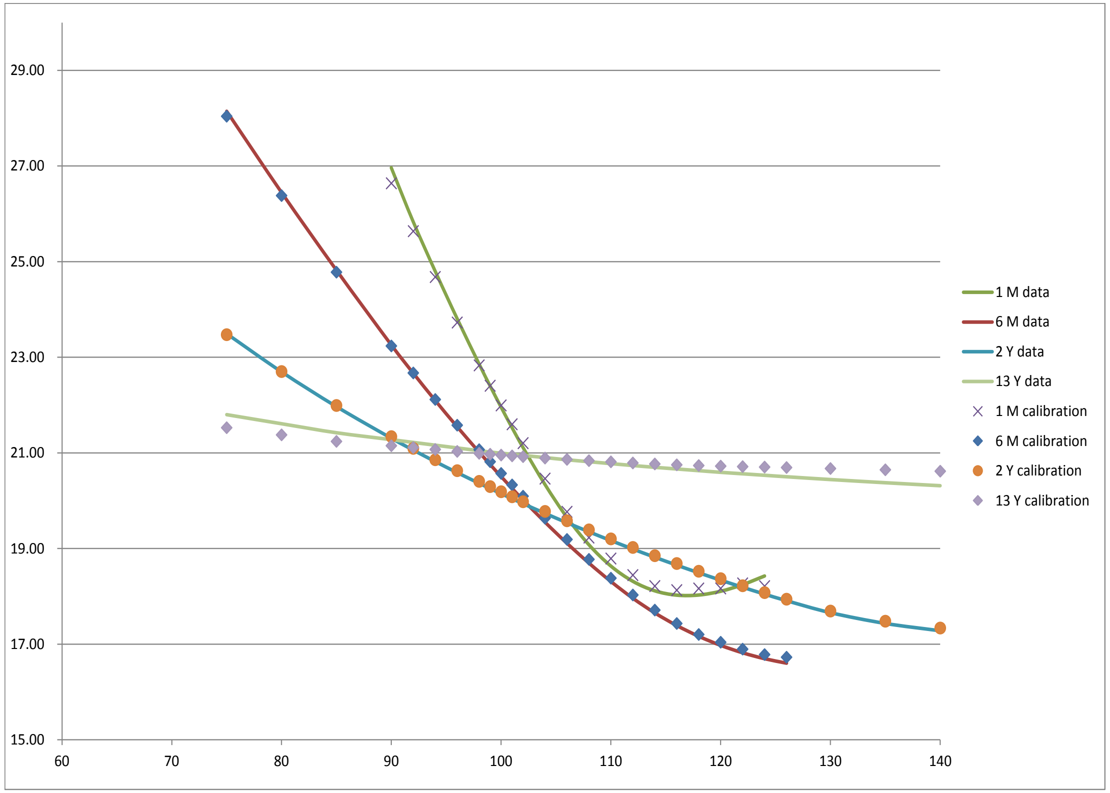

Figure: Calibrated implied volatilities compared to implied volatilities (DAX, 5-May-2015), $J = 0.9$, $\sigma = 100\%$, $\rho = 0.4$. We used 10000 particles to compute the local volatility, 40000 simulations to reprice vanillas.

Numerical implementation: $\rho = -0.4$

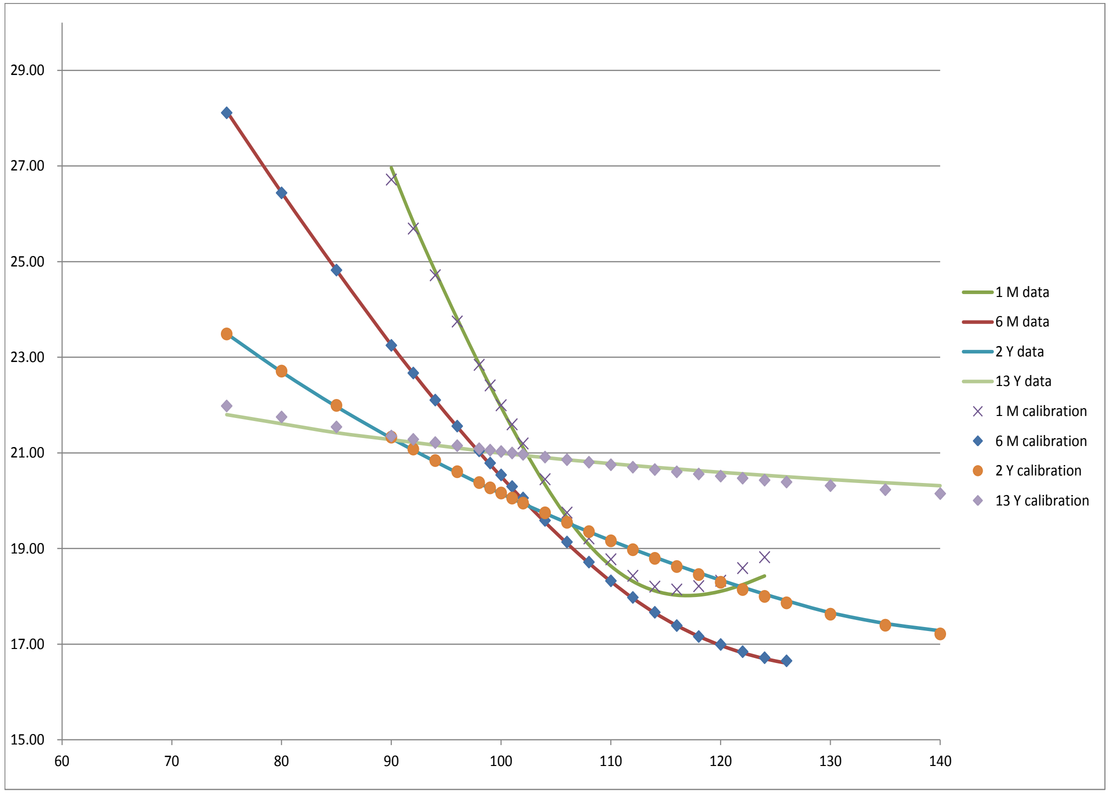

Figure: Calibrated implied volatilities compared to implied volatilities (DAX, 5-May-2015), $J = 0.9$, $\sigma = 100\%$, $\rho = -0.4$. We used 10000 particles to compute the local volatility, 40000 simulations to reprice vanillas.

Regime-switching model

SDE:

$$\frac{dS_t}{S_t} = \sigma_1(t, S_t)\, dW_t + (J-1)(dN_t - \lambda_t dt), \quad \forall t \leq \tau$$

$$\frac{dS_t}{S_t} = \sigma_2(t, S_t)\, dW_t, \quad \forall t > \tau$$

where τ is the first time to default of a Poisson process $(N_t)_{t\geq 0}$.

Calibration condition

Setting $\sigma_2(t,K) = J_S(t,K)\sigma_1(t,K)$,

Proposition 5

$S_t \sim \mathbb{P}_t^{\mathrm{mkt}}$ for all $t \leq T$ if and only if

$$\begin{aligned}\sigma_1(t,K)^2 &= \frac{\sigma_{\mathrm{Dupire}}(t,K)^2}{1+(J_S(t,K)^2-1)P_2(t,K)} \\ &+ 2\frac{(J-1)(\Lambda(t,K)-K\partial_K\Lambda(t,K))}{\partial_{KK}C^{\mathrm{mkt}}(t,K)(1+(J_S(t,K)^2-1)P_2(t,K))} \\ &- 2\frac{J\Lambda(t,\frac{K}{J})-\Lambda(t,K)}{\partial_{KK}C^{\mathrm{mkt}}(t,K)(1+(J_S(t,K)^2-1)P_2(t,K))}\end{aligned}$$

where: $P_2(t,K) := \mathbb{E}[\mathbf{1}_{\tau<t}|S_t = K]$ and
$\Lambda(t,K) := \mathbb{E}\left[\lambda_t \mathbf{1}_{\tau>t}(S_t-K)_+\right]$.

Some simplifications

- A conditioning argument:

$$\begin{aligned}\Lambda(t,K) &= \mathbb{E}[\lambda_t e^{-\int_0^t \lambda_s ds}(S_t^1 - K)_+], \\ P_2(t,K) &= 1 - \frac{\mathbb{E}[e^{-\int_0^t \lambda_s ds}\delta(S_t^1 - K)]}{\partial_{KK} C^{\mathrm{mkt}}(t,K)},\end{aligned}$$

  where the process S_t^1 satisfies the SDE:

$$\frac{dS_t^1}{S_t^1} = \sigma_1\left(t, S_t\right) dW_t - (J-1)\lambda_t dt, \quad S_0^1 = S_0.$$

- The regularized process $S_t^{1,\epsilon}$ (obtained by regularizing $\delta(S_t^1 - K)$ and $(S_t^1 - K)_+$) exists.

Numerical computations

- **Algorithm** Divide the interval $[0, T]$ into intervals of size Δ.
  - $t := 0$, set $\sigma(t, S) = \sigma_{\text{Dupire}}(t, S)$ between 0 and Δ and diffuse the N particles up to Δ.
  - Compute $\Lambda(\Delta, K)$ using Monte Carlo by:

    $$\Lambda(t, K) = \frac{1}{N} \sum_{i=1}^{N} \lambda_t^i e^{-\int_0^t \lambda_s^i ds} (S_t^{i,1} - K)_+$$

    and

    $$P_2(t, K) = 1 - \frac{\sum_{i=1}^{N} e^{-\int_0^t \lambda_s^i ds} \delta_{t,N}(S_t^{i,1} - K)}{N \partial_{KK} C^{\text{mkt}}(t, K)}$$

    Then, compute the local volatility $\sigma(\Delta, S)$ for all S using Equation (2), and set $\sigma(t, S) = \sigma(\Delta, S)$ for all $t \in [\Delta, 2\Delta]$.
  - Iterate up to maturity T.
- Take $\delta_{t,N}(x) = \frac{1}{h_{t,N}} K(\frac{x}{h_{t,N}})$, the kernel $K(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)$, $h_{t,N} = \kappa S_0 \sigma_0 \sqrt{\max(t, t_{\min})} N^{-1/5}$ ($\kappa = 1.5$, $t_{\min} = 1/4$, and $\sigma_0 = 20\%$).

Numerical implementation: $\rho = 0$

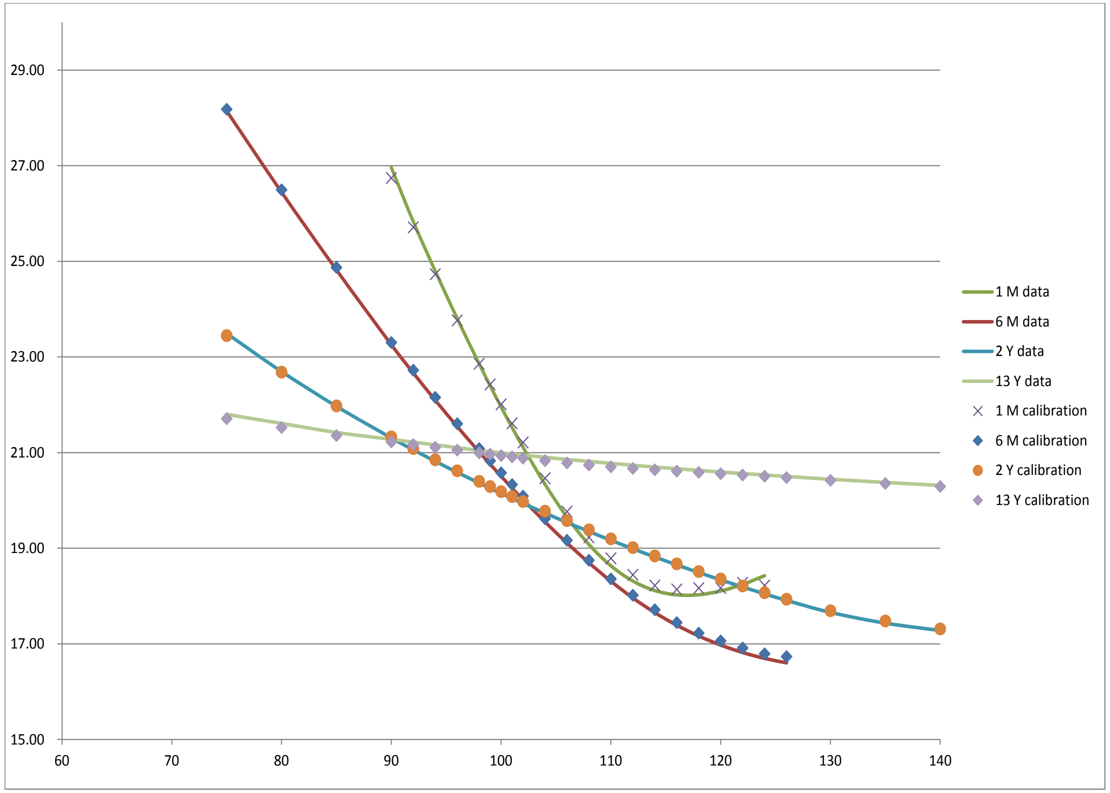

Figure: Calibrated implied volatilities compared to implied volatilities (DAX, 5-May-2015), $J = 0.9$, $J_S = 1.2$, $\sigma = 100\%$, $\rho = 0$. We used 1000 particles, 40000 simulations.

Numerical implementation: $\rho = 0.4$

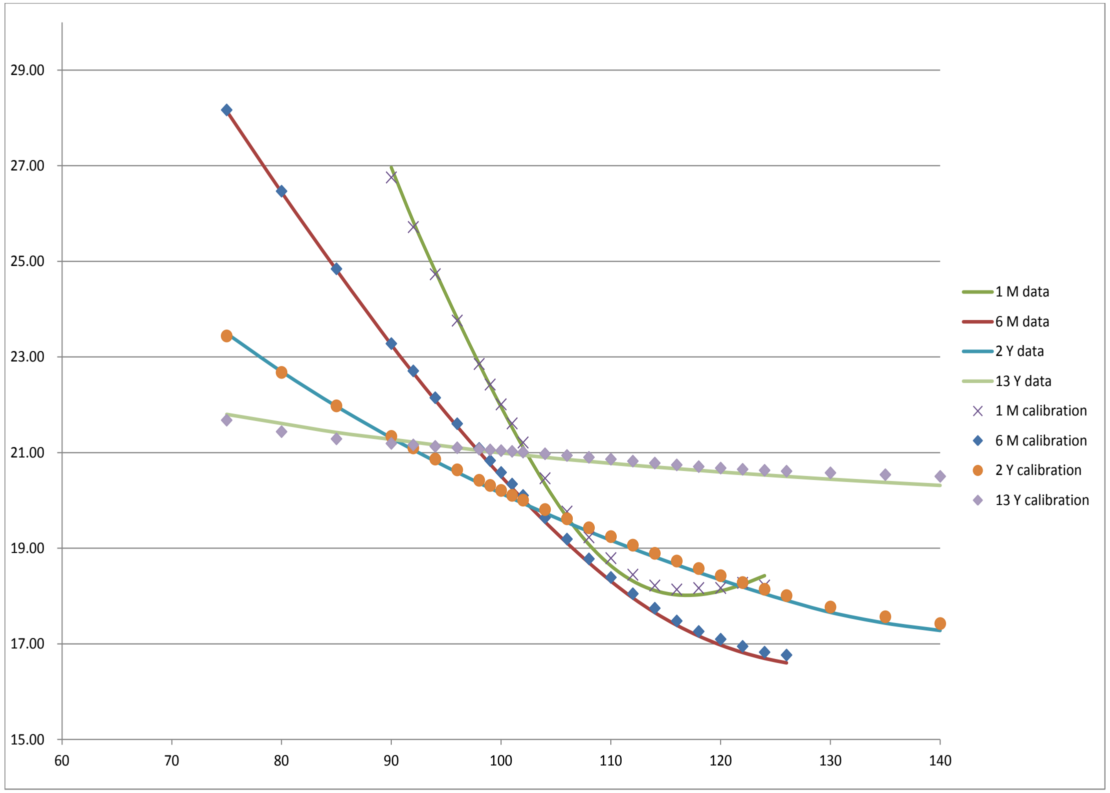

Figure: Calibrated implied volatilities in regime switching model compared to implied volatilities (DAX, 5-May-2015), $J = 0.9$, $J_S = 1.2$, $\sigma = 100\%$, $\rho = 0.4$. We used 10000 particles, 40000 simulations.

Numerical implementation: $\rho = -0.4$

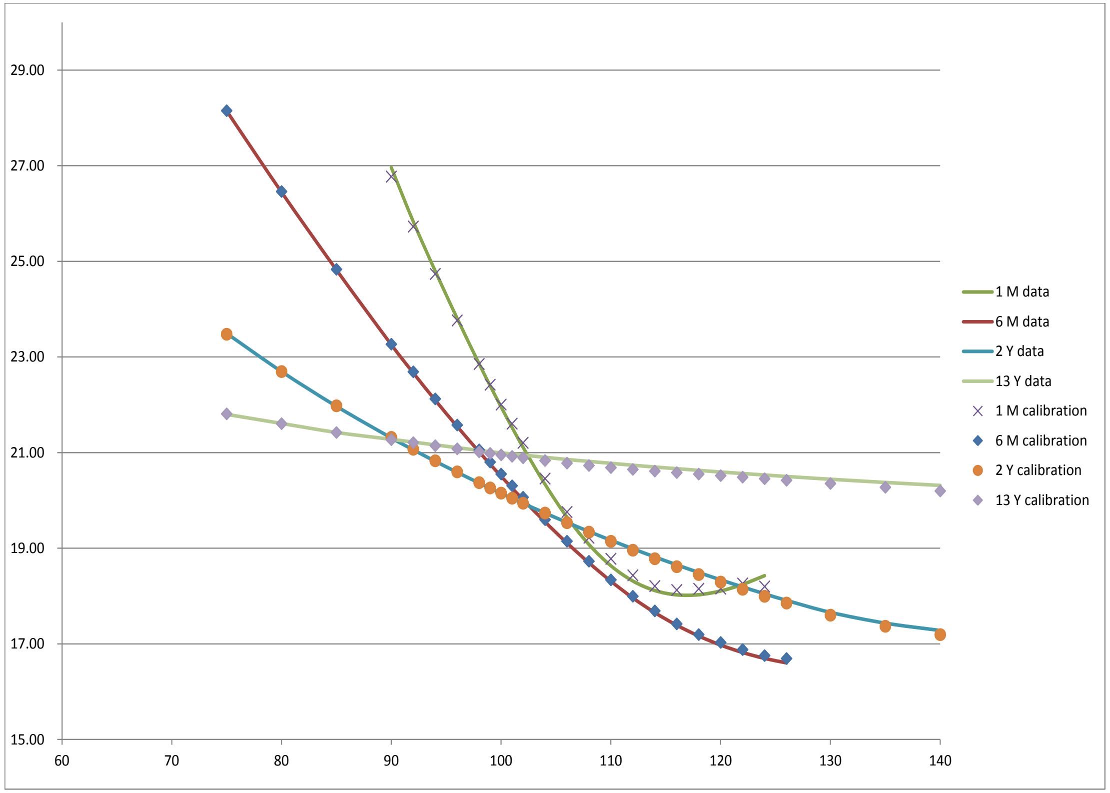

Figure: Calibrated implied volatilities in regime switching model compared to implied volatilities (DAX, 5-May-2015), $J = 0.9$, $J_S = 1.2$, $\sigma = 100\%$, $\rho = -0.4$. We used 10000 particles, 40000 simulations.