Hörmander's condition for delayed SDEs

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Notations

- We are interested in the following stochastic differential equation on $[0,T]$,

$$X_t = X_0 + \int_0^t \sum_{k=0}^m V_k(r,X) \circ dW_r^k$$
$$= X_0 + \int_0^t \sum_{k=0}^m V_k(X_r, X_{r-h_1}, \dots, X_{r-h_{N-1}}) \circ dW_r^k$$

- for some smooth functions $V_k : (\mathbb{R}^d)^N \to \mathbb{R}^d$ and $0 = h_0 < h_1 < \cdots < h_{N-1} < T$ fixed,
- $X_t \in \mathbb{R}^d$, and $\{W_t^k\}_{t\geq 0, k=1,\dots,m}$ is a m-dimensional Brownian motion $W_t^0 = t$.

Existence and uniqueness of the solutions of the SDE

Assumption (Standing regularity assumption on V_k)

For all $k = 0, \ldots, m$, the mappings $V_k : \mathbb{R}^{d\times N} \to \mathbb{R}^d$ are assumed to be smooth with bounded derivatives at all order.

Proposition

Under the standing assumptions X_t exists for $t \geq 0$ and

$$\mathbb{E}\left[\|X\|^p_{\infty,[0,t]}\right] < \infty, \text{ for all } t > 0,\ p \geq 2.$$

Proof by Picard's iteration with the norm

$$\|X\|_{\infty,[0,t]} := \sup_{s\in[0,t]} |X_s|$$

and Grönwall's inequality.

Objectives

- Sufficient condition so that the distribution of X_T is absolutely continuous with respect to the Lebesgue measure.
- Sufficient condition so that the density of X_T with respect to the Lebesgue measure is smooth.
- Adapt the arguments of the Markovian case to take into account the noise coming from the delay.
- $\implies$ Hörmander-like spanning condition of $\mathbb{R}^d$.

Objectives

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- Adapt the arguments of the Markovian case to take into account the noise coming from the delay.
- $\implies$ Hörmander-like spanning condition of $\mathbb{R}^d$.
- Uniformly elliptic and path-dependent case was treated by Kusuoka-Stroock(1982) and Bally-Caramellino (2016).
- Bell-Mohammed (1992), $V(X_{r-h})$ can be degenerate at X_0, but assumption on $X_{[-T,0]}$ to compensate it.

Malliavin's derivative

- X_T is Malliavin differentiable (Kusuoka-Stroock).
- Define

$$T_h := h_{N-1} \vee \sup_{i=1\ldots N-1} (T - (h_i - h_{i-1})) \in (0, T).$$

- The Malliavin derivative $\mathcal{D}_t X_T$ satisfies for all $T_h \leq t \leq T$,

$$\mathcal{D}_t X_T = V(t, X) + \sum_{k=0}^{m} \int_t^T \partial_0 V_k(r, X) \mathcal{D}_t X_r \circ dW_r^k$$

where $\partial_i V_k(r, \mathbf{x}) \in \mathbb{R}^{d \times d}$ is Jacobian of V_k in the ith component.

Sufficient condition on the Malliavin Matrix

- Let $\mathcal{M} := \int_0^T \mathcal{D}_s X_T (\mathcal{D}_s X_T)^* ds$ be the Malliavin matrix.
- Summary of the Malliavin calculus for regularity of laws: If $\forall p > 0,\ \mathbb{E}\left(\|\mathcal{M}^{-1}\|^p\right) < \infty$ then X_T has density with respect to the Lebesgue measures and its density is smooth.

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- Sufficient condition : There exist $\delta, q > 0$, $\mathcal{R} > 0$ with $\mathbb{E}[\mathcal{R}^p] < \infty$ for all $p > 1$ such that for all $\eta \in \mathbb{R}^d$ with $|\eta| = 1$ we have
$$\delta \leq \mathcal{R}\langle \eta, \mathcal{M}\eta \rangle^q.$$
- Estimate from below $\int_0^T |\eta^* \mathcal{D}_s X_T|^2_{\mathbb{R}^d}\, ds$ with $|\eta| = 1$ fixed.

Rough Paths

- Denote $\mathcal{W}_t = (W_t, W_{t-h_1}, \dots, W_{t-h_{N-1}}) \in (\mathbb{R}^d)^N$ and we fix $\frac{1}{3} < \alpha < \frac{1}{2} < \theta < 2\alpha$.
- We need to define $\mathbb{W}^{i,j}_{s,t} := \int_s^t W_{r-h_i,s-h_i} \otimes \circ dW_{r-h_j}$. If $h_i \geq h_j$. This is the classical Stratonovich integral. For $h_i < h_j$, the anticipative integral can be deduced from the first order condition

$$\mathbb{W}^{i,j}_{s,t} + \mathbb{W}^{j,i}_{s,t} = \mathcal{W}_{s-h_i,t-h_i} \otimes \mathcal{W}_{s-h_j,t-h_j}.$$

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- One can also equivalently appeal to Nualart-Pardoux (88) and Nualart-Ocone (89) where an anticipative Stratonovich integral is defined.
- $(\mathcal{W}, \mathbb{W}) \in \mathcal{C}^\alpha$, the set of α-Holder continuous rough paths (with iterated integrals in $\mathcal{C}^{2a}$).

SDE as Rough integral

- We say that (Y, Y') is a controlled path if

$$Y_t^{k,i} - Y_s^{k,i} = \sum_{k'=1}^{m} \sum_{i'=0}^{N-1} Y_s^{k,i,k',i'} \mathcal{W}_{s,t}^{k',i'} + O(|t-s|^{2\alpha}).$$

- We define the rough integral of a controlled rough path Y

$$\sum_{k,i} \int Y_r^{k,i} d\mathcal{W}_r^{k,i} := \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}} Y_u \mathcal{W}_{u,v} + Y'_u \mathbb{W}_{u,v}.$$

- For all $t \in [T_h, T]$,

$$X_t = X_0 + \int_0^t \sum_{k=0}^{m} V_k(r, X_r, X_{r-h_1}, \ldots, X_{r-h_{N-1}}) d\mathcal{W}_r^{k,0}$$

$$J_{t,T} = \frac{d\text{"} X_T \text{"}}{d\text{"} X_t \text{"}} = Id + \sum_{k=0}^{m} \int_t^T \partial_0 V_k(r, X) J_{t,r} d\mathcal{W}_r^{k,0}$$

Integration with controlled rough paths

- In particular

$$\mathcal{D}_t X_T = J_{t,T} V(t,X), \quad J_{t,T} = J_{T_h,T} J^{-1}_{T_h,t} \text{ and}$$

$$J^{-1}_{T_h,t} = Id - \sum_{k=0}^{m} \int_{T_h}^{t} J^{-1}_{T_h,r} \partial_0 V_k(r,X) d\mathcal{W}^{k,0}_r.$$

- Denote $Z_{V_j}(t) = \eta^* J_{t,T} V_j(t,X)$, we want lower bounds for

$$\begin{aligned} \mathcal{M} &:= \sum_{j=1}^{m} \int_0^T |\eta^* J_{s,T} V_j(s,X)|^2_{\mathbb{R}^d}\, ds = \sum_{j=1}^{m} \|Z_{V_j}\|^2_{L^2[0,T]} \\ &\geq \sum_{j=1}^{m} \|Z_{V_j}\|^2_{L^2[T_h,T]} \end{aligned}$$

First estimate

- Interpolation inequality

$$\sup_{s\in[T_h,T]} \left|Z_{V_j}(s)\right| \leq C_{h,T} \|Z_{V_j}\|_{L^2([T_h,T])}^{\frac{2\alpha}{2\alpha+1}} \|Z_{V_j}\|_{\alpha,[T_h,T]}^{\frac{1}{2\alpha+1}}$$

$$\leq C_{h,T} \|Z_{V_j}\|_{L^2([0,T])}^{\frac{2\alpha}{2\alpha+1}} \|Z_{V_j}\|_{\alpha,[T_h,T]}^{\frac{1}{2\alpha+1}} .$$

- A first estimate

$$\sup_{j=1\dots m} \|Z_{V_j}\|_{\infty,[T_h,T]} \leq C_{h,T} \mathcal{M}^{\frac{2\alpha}{2\alpha+1}} \sum_{j=1}^{m} \|Z_{V_j}\|_{\alpha,[T_h,T]}^{\frac{1}{2\alpha+1}}.$$

- $L^\infty([T_h,T])$ estimates for Z_{V_j}.

Brackets

- By Ito's formula, for all $j = 1, \dots, m$ and $s, t \in [T_h, T]$

$$V_j(t,X) - V_j(s,X) = \sum_{k=0}^{m} \int_s^t \partial_0 V_j(r,X) V_k(r,X) d\mathcal{W}_r^{k,0}$$
$$+ \sum_{\substack{0 \le k \le m \\ 1 \le i \le N-1}} \int_s^t \partial_i V_j(r,X) V_k(r - h_i, X) d\mathcal{W}_r^{k,i}.$$

- To define the last term one needs to extends the $\mathcal{W}$ to take into account $t - (h_i + h_j)$ terms in the control of the integrands.
- We define the Bracket

$$[V_j, V_k] := \partial_0 V_j V_k - \partial_0 V_k V_j.$$

Evolution of Z_{V_j}

- For all $s, t \in [T_h, T]$

$$Z_{V_j}(t) - Z_{V_j}(s) = \sum_{k=1}^{m} \int_s^t Z_{[V_j, V_k]}(r) d\mathcal{W}_r^{k,0}$$
$$+ \sum_{k=1}^{m} \sum_{i=1}^{N} \int_s^t Z_{\partial_i V_j(\cdot, X) V_k(\cdot - h_i, X)}(r) d\mathcal{W}_r^{k,i}$$
$$+ \int_s^t Z_{\{[V_j, V_0] + \sum_{i=1}^{N-1} \partial_i V_j(\cdot, X) V_0(\cdot - h_i, X)\}}(r) dr.$$

- $\{d\mathcal{W}_r^{k,i} : r \in (T_h, T), k = 1 \dots m, i = 0, \dots, N-1\}$ are the increments of a $m \times N$-dimensional Brownian Motion.

θ-Holder Roughness

Definition

$W : [0,T] \to \mathbb{R}^m$ is θ-Holder rough : there exists $L > 0$ such that for any $\varphi \in \mathbb{R}^m$, $s \in [0,T]$ and $\varepsilon \in (0,1)$, there exists $t \in [0,T]$ such that

$$|t-s| \leq \varepsilon, \text{ and } |\varphi W_{s,t}| \geq L\varepsilon^{\theta}|\varphi|.$$

Proposition (Norris' Lemma)

There exists a deterministic constant $C_T > 0$ and $p, q > 0$ s.t. if

$$Z_t = \int_0^t A_s dW_s + \int_0^t B_s ds, \text{ with } A, B \text{ controlled rough paths}$$

then $\|A\|_{\infty,[0,T]} + \|B\|_{\infty,[0,T]} \leq C_T \left(\frac{\mathcal{R}}{L_\theta}\right)^p \|Z\|^q_{\infty,[0,T]}$.

A variant of Norris Lemma

Proposition

The path $\mathcal{W} : [T_h, T] \to \mathbb{R}^{m\times N}$ is θ-Holder rough.

Proposition (Variant of Norris' Lemma)

There exists a deterministic constant $C_T > 0$ and $p, q > 0$ such that

$$\|Z_{[V_j,V_k]}\|_{\infty,[T'_h,T]} + \|Z_{\partial_i V_j(\cdot,X)V_k(\cdot-h_i,X)}\|_{\infty,[T'_h,T]}$$
$$+ \|Z_{\{[V_j,V_0]+\sum_{i=1}^{N-1}\partial_i V_j(\cdot,X)V_0(\cdot-h_i,X)\}}\|_{\infty,[T'_h,T]}$$
$$\leq C_{h,T}\mathcal{M}^{q_1}\left(\frac{\mathcal{R}}{L_\theta}\right)^{p_1}.$$

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Proposition

The path $\mathcal{W} : [T_h, T] \to \mathbb{R}^{m\times N}$ is θ-Holder rough.

Proposition (Variant of Norris' Lemma)

There exists a deterministic constant $C_T > 0$ and $p, q > 0$ such that

$$\|Z_{[V_j,V_k]}\|_{\infty,[T_h',T]} + \|Z_{\partial_i V_j(\cdot,X)V_k(\cdot-h_i,X)}\|_{\infty,[T_h',T]}$$
$$+ \|Z_{\{[V_j,V_0]+\sum_{i=1}^{N-1}\partial_i V_j(\cdot,X)V_0(\cdot-h_i,X)\}}\|_{\infty,[T_h',T]}$$
$$\leq C_{h,T}\mathcal{M}^{q_1}\left(\frac{\mathcal{R}}{L_\theta}\right)^{p_1}.$$

- Only $Z_{[V_j,V_k]}$ is controlled by $\mathcal{W}$.

Link between Hörmander condition and the Malliavin derivatives

Definition

We define the family of hyperplanes generated by the vector fields,

$$\mathcal{V}_0 := \{(s,\mathbf{x}) \to V_k(s,\mathbf{x}) : k = 1,\cdots,m\} \text{ and}$$

$$\mathcal{V}_{j+1} := \mathcal{V}_j \cup \{[F,V_k] : F \in \mathcal{V}_j, k = 1,\cdots,m\} \text{ and}$$

We also define the extension of the hyperplanes

$$\overline{\mathcal{V}}_{j+1} := \mathcal{V}_{j+1} \bigcup_{F\in\mathcal{V}_j} \{(s,\mathbf{x}) \to [F,V_0](s,\mathbf{x}) + \sum_{i=1}^{N-1} \partial_i F(s,\mathbf{x}) V_0(s-h_i,\mathbf{x})\}$$

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$$\begin{aligned}\overline{\mathcal{V}}_{j+1} \quad := \quad & \mathcal{V}_{j+1} \bigcup_{F\in\mathcal{V}_j} \{(s,\mathbf{x}) \to [F,V_0](s,\mathbf{x}) + \sum_{i=1}^{N-1} \partial_i F(s,\mathbf{x}) V_0(s-h_i,\mathbf{x})\} \\ & \bigcup_{F\in\mathcal{V}_j} \{(s,\mathbf{x}) \to \partial_i F(s,\mathbf{x}) V_k(s-h_i,\mathbf{x}) : \ k = 1\cdots m, \ i = 1,\ldots,N-1\}.\end{aligned}$$

Hormander

- V_{j+1} is smaller than its Markovian counter part.
- $\overline{V}_{j+1}$ includes the Brackets with V_0.
- Additional noise coming from the delay.

Assumption (Hörmander Condition for Delayed equations)

There exists j_0 such that $\overline{\mathcal{V}}_{j_0}$ spans $\mathbb{R}^d$ uniformly

$$\inf_{\mathbf{x}\in C([0,T])} \inf_{|\eta|=1} \sup_{F\in\overline{V}_{j_0}} |\eta^* F(T,\mathbf{x})| > 0$$

- Spanning uniform in $\mathbf{x}$ and in the rank j_0.

Main theorem

Theorem (Main theorem)

Under the standing regularity assumption of $\{V_k\}$ and the Hörmander condition X_T has a smooth density with respect to the Lebesgue measure.

- By induction one can show that

$$\begin{aligned} 0 < \delta := \inf_{\mathbf{x} \in C([0,T])} \inf_{|\eta|=1} \sup_{F \in \overline{V}_{j_0}} |\eta^* F(T, \mathbf{x})| \\ \leq \inf_{|\eta|=1} \sup_{F \in \overline{V}_{j_0}} |\eta^* F(T, X)| \\ \leq C_{j_0,h,T} \frac{\mathcal{R}^{p_{j_0}}}{L^{p_{j_0}}} \mathcal{M}^{q_{j_0}}. \end{aligned}$$

Unifomly non-degenerate

- Assume that there exists $\delta > 0$ such that for all $|\eta| = 1$

$$\sum_{k=1}^{m} |\eta^* V_k|^2 \geq \delta$$

  Then Hörmander's condition for delayed SDEs is satisfied.
- This case was treated by Kusuoka-Stroock(1982) and Bally-Caramellino (2016) without the assumption on the delay.

Langevin Equation with Delay

Consider the diffusion in $\mathbb{R}^2$,

$$\begin{aligned} dp_t &= V_0(p_t, q_t)dt + V_1(p_t, q_t, p_{t-h}, q_{t-h}) \circ dW_t \\ dq_t &= p_t dt. \end{aligned}$$

with V_1 uniformly elliptic. We check the spanning condition

$$\mathcal{V}_0 = \left\{ \begin{pmatrix} V_1 \\ 0 \end{pmatrix} \right\}$$

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with V_1 uniformly elliptic. We check the spanning condition

$$\mathcal{V}_0 = \left\{ \begin{pmatrix} V_1 \\ 0 \end{pmatrix} \right\} = \mathcal{V}_j$$

We compute $\overline{\mathcal{V}}_1$

$$\left[\begin{pmatrix} V_1 \\ 0 \end{pmatrix}, \begin{pmatrix} V_0 \\ p_s \end{pmatrix} \right] + \partial_1 \begin{pmatrix} V_1 \\ 0 \end{pmatrix} \begin{pmatrix} V_0(p_{s-h}, q_{s-h}) \\ 0 \end{pmatrix} = \begin{pmatrix} * \\ -V_1 \end{pmatrix}$$

$\Longrightarrow$ uniform spanning

Noise from the delay

We now consider the following diffusion

$$\begin{pmatrix} p_t \\ q_t \\ r_t \end{pmatrix} = \int_0^t \begin{pmatrix} 1 \\ 1 \\ -r_{s-h} \end{pmatrix} dW_s^1 + \int_0^t \begin{pmatrix} -p_{s-h} \\ q_{s-h} \\ \sqrt{1 + r_{s-h}^2} \end{pmatrix} dW_s^2$$

We check the spanning condition

$$\mathcal{V}_0 = \left\{ \begin{pmatrix} 1 \\ 1 \\ -r_{s-h} \end{pmatrix}, \begin{pmatrix} -p_{s-h} \\ q_{s-h} \\ \sqrt{1 + r_{s-h}^2} \end{pmatrix} \right\}$$

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We compute the semi-brackets $\partial_1 V_2(t) V_1(t-h)$, $\partial_1 V_1(t) V_2(t-h)$ hence the subset of $\overline{\mathcal{V}}_0$:

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ -r_{s-h} \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -\frac{r_{s-h} r_{s-2h}}{\sqrt{1+r_{s-h}^2}} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -\sqrt{1+r_{s-2h}^2} \end{pmatrix} \right\} \subset \overline{\mathcal{V}}_0$$

$\Longrightarrow$ uniform spanning

THANK YOU!

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ALLEZ LES BLEUS