# Nonzero-sum stochastic differential games with impulse controls and applications to retail energy markets

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**A practical example**. In energy markets, retailers buy energy in the wholesale market and re-sell it to final customers.

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High final prices mean high incomes, but few customers; conversely, low final prices imply high market share, but low unitary incomes. Moreover, the market share also depends on the opponent's choices.

Each retailer wants to maximize his incomes: we model this competition as a two-player stochastic differential game and look for Nash equilibria in the retailers' price management policy.

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- The price management policy of player $i$ is determined by the sequence $u^i = \{(\tau_{i,k}, \delta_{i,k})\}_k$ (impulse control), where $\tau_{i,k}$ are the intervention times and $\delta_{i,k}$ are the corresponding shifts.

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- Intervening has a (fixed) cost for player $i$, denoted $c_i$. He also faces operational costs, quadratic w.r.t. his market share $\Phi^i$.
- The players' market share depends on the difference between the prices they ask: $\Phi^i_t = \Phi(P^i_t - P^j_t) \in [0, 1]$, for suitable $\Phi$. In our model, $\Phi(\eta) = \min\left\{1,\ \max\{0, -(\eta - \Delta)/(2\Delta)\}\right\}$.

So, player $i$ buys and re-sells energy ($\rightarrow$ continuous-time revenue),

$$\int_0^\infty e^{-\rho t} \;\; (P^i_t - S_t)\, \Phi\big(P^i_t - P^j_t\big)$$

So, player $i$ buys and re-sells energy ($\rightarrow$ continuous-time revenue), pays quadratic operational costs ($\rightarrow$ continuous-time spending),

$$\int_0^\infty e^{-\rho t} \left( (P_t^i - S_t)\, \Phi\big(P_t^i - P_t^j\big) - \frac{b_i}{2} \Phi\big(P_t^i - P_t^j\big)^2 \right) dt$$

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**The problem**. We look for Nash equilibria, in order to maximize the players' incomes. In particular, player $i$ wants to maximize

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To the best of our knowledge, no references are present in the literature about this class of problems.

Indeed, related works only address the following problems.

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$$\mathbb{E}\left[ \int_0^{\tau} e^{-\rho t} f(X_t) dt + e^{-\rho\tau} h(X_\tau) \right].$$

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Friedman: the players choose $\tau_1, \tau_2$ so as to maximize

$$\mathbb{E}\left[ \int_0^{\tau_1 \wedge \tau_2} e^{-\rho t} f(X_t) dt + e^{-\rho(\tau_1 \wedge \tau_2)} h(X_{\tau_1 \wedge \tau_2}) \right].$$

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Bensoussan-Friedman: the players choose $\tau_1, \tau_2$ so as to maximize

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Cosso: the players choose $u^i = \{(\tau^i_k, \delta^i_k)\}_k$ so as to maximize

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1. Rigorous formalization of the problem.
2. Verification theorem.
3. Application to competition in retail energy markets.

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- Any intervention by one of the players corresponds to a cost for the intervening player and a gain for the opponent.
- Player $i \in \{1,2\}$ wants to maximize the following payoff (running payoff, intervention costs and gains, terminal cost):

$$\mathbb{E}_x\bigg[\int_0^{\tau_S} e^{-\rho_i s} f_i(X_s)ds + \sum_k e^{-\rho_i \tau_{i,k}} \phi_i\Big(X_{(\tau_{i,k})^-}, \delta_{i,k}\Big) + \sum_k e^{-\rho_i \tau_{j,k}} \psi_i\Big(X_{(\tau_{j,k})^-}, \delta_{j,k}\Big) + e^{-\rho_i \tau_S} h_i\big(X_{(\tau_S)^-}\big) \mathbb{1}_{\{\tau_S < +\infty\}}\bigg].$$

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We now provide a rigorous formulation for such problems.

**The process.** The underlying process, when none of the player intervenes, is modelled by $dY_s = b(Y_s)ds + \sigma(Y_s)dW_s \in \mathbb{R}^d$. The game ends at $\tau_S$, the exit time of $Y$ from a fixed subset $S \subseteq \mathbb{R}^n$.

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**Interventions of the players.** When player $i \in \{1,2\}$ decides to intervene with impulse $\delta$, the process is shifted from state $y$ to state $\Gamma^i(y,\delta)$. Moreover, player $i$ pays a penalty $\phi_i(x,\delta)$ (interven. cost), whereas his opponent player $j$ earns $\psi_j(x,\delta)$ (intervention gains).

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**Impulse controls.** The action of player $i$ is modelled by a sequence (impulse control) in the form $u_i = \{(\tau_{i,k}, \delta_{i,k})\}_{k \geq 1}$, where $\{\tau_{i,k}\}_k$ are increasing stopping times (the intervention times) and $\{\delta_{i,k}\}_k$ are random variables (the corresponding impulses).

**Strategies.** The behaviour of the players, modelled by impulse controls, is driven by strategies.

A strategy for player $i \in \{1, 2\}$ is a couple $\varphi_i = (A_i, \xi_i)$, where $A_i$ is a fixed subset of $\mathbb{R}^d$ and $\xi_i$ is a continuous function.

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Once the couples $\varphi_i = (A_i, \xi_i)$ and a starting point $x$ have been chosen, a couple of impulse controls and a controlled process $X = X^{x;\varphi_1,\varphi_2}$ are uniquely defined by the following procedure:

- player $i$ intervenes if and only if the process exits from $A_i$, in which case the impulse is given by $\xi_i(y)$, where $y$ is the state;
- if both the players want to act, player 1 has the priority.

**Nash equilibria.** Let $\varphi_i = (A_i, \xi_i)$ be the strategies and $x$ be the initial state. Player $i$ aims at maximising the following functional (running payoff, intervention costs, intervention gains, final cost):

$$J^i(x; \varphi_1, \varphi_2) := \mathbb{E}_x \bigg[ \int_0^{\tau_S} e^{-\rho_i s} f_i(X_s) ds + \sum_k e^{-\rho_i \tau_{i,k}} \phi_i \Big( X_{(\tau_{i,k})^-}, \delta_{i,k} \Big) + \sum_k e^{-\rho_i \tau_{j,k}} \psi_i \Big( X_{(\tau_{j,k})^-}, \delta_{j,k} \Big) + e^{-\rho_i \tau_S} h_i \big( X_{(\tau_S)^-} \big) \mathbb{1}_{\{\tau_S < +\infty\}} \bigg].$$

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We say that a couple of strategies $(\varphi_1^*, \varphi_2^*)$ is a Nash equilibrium if

$$V^1(x) := J^1(x;\varphi_1^*,\varphi_2^*) \geq J^1(x;\varphi_1,\varphi_2^*), \qquad \forall \varphi_1,$$
$$V^2(x) := J^2(x;\varphi_1^*,\varphi_2^*) \geq J^2(x;\varphi_1^*,\varphi_2), \qquad \forall \varphi_2.$$

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**Heuristics on $\varphi_i^*$.** Assume we know $V_i$ and that there exists $\delta_i$ s.t.

$$\{\delta_i(x)\} = \arg\max_\delta \big(V_i\big(x+\delta\big) + \phi_i(x,\delta)\big),$$

for each $i \in \{1,2\}$, $x \in S$. Then, for each $i,j \in \{1,2\}$, $i \neq j$, $x \in S$, let

$$\begin{aligned}\mathcal{M}_i V_i(x) &= V_i\big(x+\delta_i(x)\big) + \phi_i\big(x,\delta_i(x)\big),\\ \mathcal{H}_i V_i(x) &= V_i\big(x+\delta_j(x)\big) + \psi_i\big(x,\delta_j(x)\big).\end{aligned}$$

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- $\mathcal{M}_i V_i(x)$ (resp. $\mathcal{H}_i V_i(x)$) is the value of the game for player $i$ in case of an immediate interv. by player $i$ (resp. player $j$).

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*To help with the interpretation, we here recall the definitions:*

$$V_i(x) = J^i(x; \varphi_1^*, \varphi_2^*),$$

$$\{\delta_i(x)\} = \arg\max_\delta \big(V_i(x+\delta) + \phi_i(x,\delta)\big),$$

$$\mathcal{M}_i V_i(x) = V_i\big(x + \delta_i(x)\big) + \phi_i\big(x, \delta_i(x)\big),$$

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- $\delta_i(x)$ is the optimal impulse of player $i$ in case of an immediate intervention by player $i$ himself;
- $\mathcal{M}_i V_i(x)$ (resp. $\mathcal{H}_i V_i(x)$) is the value of the game for player $i$ in case of an immediate interv. by player $i$ (resp. player $j$).

As a consequence, we (heuristically) argue that the Nash policy is:

*player $i$ intervenes if and only if $\mathcal{M}_i V_i(x) = V_i(x)$*
*and shifts the process from $x$ to $x + \delta_i(x)$.*

Indeed, the verification theorem will make this guess rigorous. But we first need to characterize $V_i$, by means of suitable equations.

**Heuristics on $V_i$.** We consider the following quasi-variational inequalities (QVI) for $V_1$ and $V_2$, where $i, j \in \{1, 2\}$ and $i \neq j$:

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$$V_i = h_i, \qquad \text{in } \partial S,$$

*First equation*. Standard terminal condition.

**Heuristics on $V_i$.** We consider the following quasi-variational inequalities (QVI) for $V_1$ and $V_2$, where $i, j \in \{1, 2\}$ and $i \neq j$:

$$\begin{aligned} &V_i = h_i, && \text{in } \partial S, \\ &\mathcal{M}_j V_j - V_j \leq 0, && \text{in } S, \end{aligned}$$

*Second equation*. We expect $\mathcal{M}_j V_j - V_j \leq 0$ thanks to the interpretation above.

**Heuristics on $V_i$.** We consider the following quasi-variational inequalities (QVI) for $V_1$ and $V_2$, where $i, j \in \{1, 2\}$ and $i \neq j$:

$$
\begin{aligned}
&V_i = h_i, && \text{in } \partial S,\\
&\mathcal{M}_j V_j - V_j \leq 0, && \text{in } S,\\
&\mathcal{H}_i V_i - V_i = 0, && \text{in } \{\mathcal{M}_j V_j - V_j = 0\},
\end{aligned}
$$

*Third equation*. If player $j$ intervenes (i.e. $\mathcal{M}_j V_j - V_j = 0$), by the definition of Nash equilibrium we expect that player $i$ does not lose anything: this is modelled by $\mathcal{H}_i V_i - V_i = 0$.

**Heuristics on $V_i$.** We consider the following quasi-variational inequalities (QVI) for $V_1$ and $V_2$, where $i, j \in \{1, 2\}$ and $i \neq j$:

$$
\begin{array}{ll}
V_i = h_i, & \text{in } \partial S, \\
\mathcal{M}_j V_j - V_j \leq 0, & \text{in } S, \\
\mathcal{H}_i V_i - V_i = 0, & \text{in } \{\mathcal{M}_j V_j - V_j = 0\}, \\
\max\left\{\mathcal{A} V_i - \rho_i V_i + f_i, \mathcal{M}_i V_i - V_i\right\} = 0, & \text{in } \{\mathcal{M}_j V_j - V_j < 0\},
\end{array}
$$

where $\mathcal{A} V_i = b \cdot \nabla V_i + \mathrm{tr}(\sigma\sigma^t D^2 V_i)/2$ (infinitesimal generator).

*Fourth equation*. If player $j$ does not intervene (i.e. $\mathcal{M}_j V_j - V_j < 0$), then $V_i$ satisfies the PDE of a standard one-player impulse problem.

**Heuristics on $V_i$.** We consider the following quasi-variational inequalities (QVI) for $V_1$ and $V_2$, where $i, j \in \{1, 2\}$ and $i \neq j$:

$$
\begin{aligned}
&V_i = h_i, && \text{in } \partial S,\\
&\mathcal{M}_j V_j - V_j \leq 0, && \text{in } S,\\
&\mathcal{H}_i V_i - V_i = 0, && \text{in } \{\mathcal{M}_j V_j - V_j = 0\},\\
&\max\left\{\mathcal{A}V_i - \rho_i V_i + f_i, \mathcal{M}_i V_i - V_i\right\} = 0, && \text{in } \{\mathcal{M}_j V_j - V_j < 0\},
\end{aligned}
$$

where $\mathcal{A}V_i = b \cdot \nabla V_i + \mathrm{tr}(\sigma\sigma^t D^2 V_i)/2$ (infinitesimal generator).

**Statement and proof.** We are now ready to state and prove the verification theorem for our class of problems.

## Verification theorem

Let $V_1, V_2$ be functions from $S$ to $\mathbb{R}$ satisfying some (very weak) technical assumptions and such that:

- $V_i$ is a classical solution to (QVI),
- $V_i \in C^2(D_j \setminus \partial D_i) \cap C^1(D_j) \cap C(S)$ and has polyn. growth,

where $i, j \in \{1, 2\}$ and $D_i = \{\mathcal{M}_i V_i - V_i < 0\}$. Let $x \in S$ and let

$$\varphi_i^* = (D_i, \delta_i).$$

Assume that $(\varphi_1^*, \varphi_2^*)$ is admissible; then,

$$(\varphi_1^*, \varphi_2^*) \text{ is a Nash equilibrium,}$$
$$V_i(x) = J^i(x; \varphi_1^*, \varphi_2^*), \text{ for } i \in \{1, 2\}.$$

### Verification theorem (practical version)

Let $V_1, V_2$ be functions from $S$ to $\mathbb{R}$ satisfying some (very weak) technical assumptions and such that:

- $V_i$ is a classical solution to (QVI),
- $V_i \in C^2(D_j \setminus \partial D_i) \cap C^1(D_j) \cap C(S)$ and has polyn. growth,

where $i, j \in \{1, 2\}$ and $D_i = \{\mathcal{M}_i V_i - V_i < 0\}$. Then $V_1, V_2$ are the value functions and a Nash equilibrium is as follows.

- Player $i$ intervenes if and only if $X$ exits from $\{\mathcal{M}_i V_i - V_i < 0\}$.
- When intervening, player $i$ shifts $X$ from the current state $x$ to the state $x + \delta_i(x)$, where $\delta_i(x)$ is the (unique) maximizer of $\delta \mapsto V_i(x + \delta) + \phi_i(x, \delta)$.

**The problem.** Let us come back to the initial problem.

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- Two retailers buy energy at price $S_t = s + \mu t + \sigma W_t$ and re-sell it at (piecewise constant) price $P^i_t = p^i + \sum_{\tau_{i,k} \leq t} \delta_{i,k}$.
- Each intervention to adjust the price costs $c_i$ to player $i$. Also, operational costs, quadratic w.r.t. his market share $\Phi(P^i_t - P^j_t)$.
- Payoff: continuous gain (sale of energy), continuous spending (operational costs), discrete spending (intervention costs).

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- Payoff: continuous gain (sale of energy), continuous spending (operational costs), discrete spending (intervention costs).

Nonzero-sum impulsive game where player $i$ wants to maximize (three-dimensional problem)

$$\mathbb{E}\left[ \int_0^\infty e^{-\rho t} \left( \left(P^i_t - S_t\right) \Phi\left(P^i_t - P^j_t\right) - \frac{b_i}{2} \Phi\left(P^i_t - P^j_t\right)^2 \right) dt - \sum_{k \geq 1} e^{-\rho \tau_{i,k}} c_i \right].$$

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- Payoff: continuous gain (sale of energy), continuous spending (operational costs), discrete spending (intervention costs).

Nonzero-sum impulsive game where player $i$ wants to maximize (~~three-dimensional~~ two-dimensional problem, with $X^i = P^i - S$)

$$\mathbb{E}\left[ \int_0^\infty e^{-\rho t} \left( X^i_t \, \Phi(X^i_t - X^j_t) - \frac{b_i}{2} \Phi(X^i_t - X^j_t)^2 \right) dt - \sum_{k \geq 1} e^{-\rho \tau_{i,k}} c_i \right].$$

We now apply the verif. theorem to characterize the Nash equilibria.

- Step 1: we solve the QVI problem to get a pair of (parametric) candidates $\tilde{V}_1, \tilde{V}_2$ for the value functions $V_1, V_2$.

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- Step 1: we solve the QVI problem to get a pair of (parametric) candidates $\tilde{V}_1, \tilde{V}_2$ for the value functions $V_1, V_2$.

$$\tilde{V}_1(x_1, x_2) = \begin{cases} \varphi_1\big(x_1^*(x_2), x_2\big) - c_1, & \text{in } R, \\ \varphi_1\big(x_1, x_2\big), & \text{in } W, \\ \varphi_1\big(x_1, x_2^*(x_1)\big), & \text{in } B, \end{cases}$$

$$\tilde{V}_2(x_1, x_2) = \begin{cases} \varphi_2\big(x_1^*(x_2), x_2\big), & \text{in } R, \\ \varphi_2\big(x_1, x_2\big), & \text{in } W, \\ \varphi_2\big(x_1, x_2^*(x_1)\big) - c_2, & \text{in } B. \end{cases}$$

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$R = \{P1 \text{ interv.}\} = \big\{(x_1, x_2) : x_1 \notin ]\underline{x}_1(x_2), \bar{x}_1(x_2)[\big\}$

$B = \{P2 \text{ interv.}\} = \big\{(x_1, x_2) : x_1 \in ]\underline{x}_1(x_2), \bar{x}_1(x_2)[,\ x_2 \notin ]\underline{x}_2(x_1), \bar{x}_2(x_1)[\big\}$

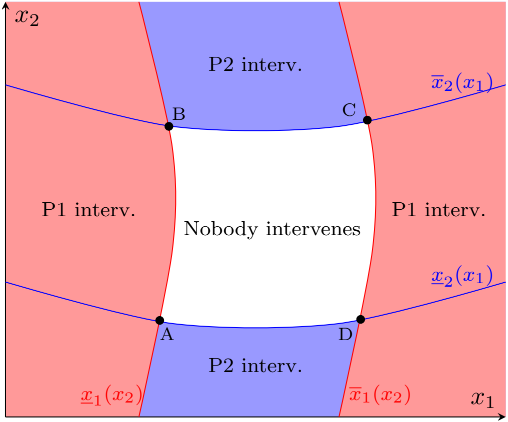
$W = \{\text{no one int.}\} = \big\{(x_1, x_2) : x_1 \in ]\underline{x}_1(x_2), \bar{x}_1(x_2)[, x_2 \in ]\underline{x}_2(x_1), \bar{x}_2(x_1)[\big\}$

$x_1^*(x_2)$ is a local max of $\varphi_1(\cdot, x_2)$, $x_2^*(x_1)$ is a local max of $\varphi_2(x_1, \cdot)$

$\varphi_1$ is explicitly known (depends on some parameters)

We now apply the verif. theorem to characterize the Nash equilibria.

- Step 1: we solve the QVI problem to get a pair of (parametric) candidates $\tilde{V}_1, \tilde{V}_2$ for the value functions $V_1, V_2$.



- Step 2: we impose the regularity conditions required in the verification theorem to such candidates. This corresponds to 11+11 functional equations.

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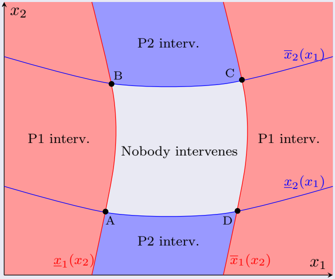
$$
\begin{cases}
\left(\frac{\partial \varphi_1}{\partial x_1}\right)\big(x_1^*(x_2), x_2\big) = 0, & x_2 \in \mathbb{R}, \\
\varphi_1\big(x_1^*(x_2), x_2\big) = \varphi_1\big(\underline{x}_1(x_2), x_2\big) + c_1, & x_2 \in [x_2^A, x_2^B], \\
\varphi_1\big(x_1^*(x_2), x_2\big) = \varphi_1\Big(\underline{x}_1(x_2), x_2^*\big(\underline{x}_1(x_2)\big)\Big) + c_1, & x_2 \in \mathbb{R} \setminus [x_2^A, x_2^B], \\
\varphi_1\big(x_1^*(x_2), x_2\big) = \varphi_1\big(\bar{x}_1(x_2), x_2\big) + c_1, & x_2 \in [x_2^D, x_2^C], \\
\varphi_1\big(x_1^*(x_2), x_2\big) = \varphi_1\Big(\bar{x}_1(x_2), x_2^*\big(\bar{x}_1(x_2)\big)\Big) + c_1, & x_2 \in \mathbb{R} \setminus [x_2^D, x_2^C], \\
\varphi_1\big(x_1, x_2^*(x_1)\big) = \varphi_1\big(x_1, \underline{x}_2(x_1)\big), & x_1 \in ]x_1^A, x_1^D[, \\
\varphi_1\big(x_1, x_2^*(x_1)\big) = \varphi_1\big(x_1, \bar{x}_2(x_1)\big), & x_1 \in ]x_1^B, x_1^C[, \\
\left(\frac{\partial \varphi_1}{\partial x_1}\right)\big(\underline{x}_1(x_2), x_2\big) = 0, & x_2 \in [x_2^A, x_2^B], \\
\left(\frac{\partial \varphi_1}{\partial x_2}\right)\big(\underline{x}_1(x_2), x_2\big) = \left(\frac{\partial \varphi_1}{\partial x_2}\right)\big(x_1^*(x_2), x_2\big), & x_2 \in [x_2^A, x_2^B], \\
\left(\frac{\partial \varphi_1}{\partial x_1}\right)\big(\bar{x}_1(x_2), x_2\big) = 0, & x_2 \in [x_2^D, x_2^C], \\
\left(\frac{\partial \varphi_1}{\partial x_2}\right)\big(\bar{x}_1(x_2), x_2\big) = \left(\frac{\partial \varphi_1}{\partial x_2}\right)\big(x_1^*(x_2), x_2\big), & x_2 \in [x_2^D, x_2^C].
\end{cases}
$$

- Step 2: we impose the regularity conditions required in the verification theorem to such candidates. This corresponds to 11+11 functional equations.

- Conclusions. If a sol. exists, the Nash equilibrium is as follows.

1. Player $i$ intervenes if and only if the state variable $(X_t^1, X_t^2)$ touches the boundary of his continuation region $]\underline{x}_i(X_j), \bar{x}_i(X_j)[$.

2. When this happens, he moves the state variable he controls, i.e. $X_t^i$, to the new state $x_i^*(X_t^j)$.



Introduction

1. Non-zero-sum impulsive games

2. Verification theorem

3. Competition in retail energy markets

Conclusions

## 1. Nonzero-sum impulsive games

1.1 Naturally arise in energy finance but never studied
1.2 Our model: strategies $\rightarrow$ impulse controls $\rightarrow$ controlled process
1.3 Payoff: running cost, intervention costs, intervention gains, terminal cost

## 2. Verification theorem

2.1 Sufficient conditions to characterize the value functions
2.2 Fundamental assumptions: QVI problem + regularity conditions
2.3 Key-points for the QVI problem: operators $\mathcal{M}_i V_i$ and $\mathcal{H}_i V_i$

## 3. Competition in retail energy markets

3.1 Two competitive retailers have do decide their price management policy
3.2 Step 1: looking for a solution to the QVI problem
3.3 Step 2: applying the regularity condition to the candidate in Step 1

*Thank you!*

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# Appendix

# Competition in retail markets: complete solution

We are going to apply the verification theorem to try and characterize the value functions and the Nash equilibria.

- Step 1: we solve the QVI problem to get a pair of (parametric) candidates $\tilde{V}_1, \tilde{V}_2$ for the value functions $V_1, V_2$.
- Step 2: we impose the regularity conditions required in the verification theorem to such candidates.

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- Step 2: we impose the regularity conditions required in the verification theorem to such candidates.

**Step 1: building a candidate.** As anticipated, we start by solving the QVI problem. First, we outline some empirical arguments to guess the form of the regions where each player intervenes.

Recall the practical meaning of the new variables: $X_t^i = P_t^i - S_t$ is the net gain from the sale of energy at time $t$.

Heuristically, player 1 intervenes iff his income $X_t^1$ exits from a suitable interval $]\underline{x}_1(X_t^2), \bar{x}_1(X_t^2)[$ (clearly depending on $X_t^2$):

$$\{\textit{P1 interv.}\} = \big\{(x_1, x_2) : x_1 \notin ]\underline{x}_1(x_2), \bar{x}_1(x_2)[\big\}.$$

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Similar argument for player 2, but we exclude the points where player 1 intervenes (he has priority in case of contemporary interv.):

$$\begin{aligned}\{P2\ interv.\} &= \big\{(x_1, x_2) : x_2 \notin ]\underline{x}_2(x_1), \bar{x}_2(x_1)[\big\} \setminus \{P1\ interv.\}\\ &= \big\{(x_1, x_2) : x_1 \in ]\underline{x}_1(x_2), \bar{x}_1(x_2)[,\ x_2 \notin ]\underline{x}_2(x_1), \bar{x}_2(x_1)[\big\}.\end{aligned}$$

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Similar argument for player 2, but we exclude the points where player 1 intervenes (he has priority in case of contemporary interv.):

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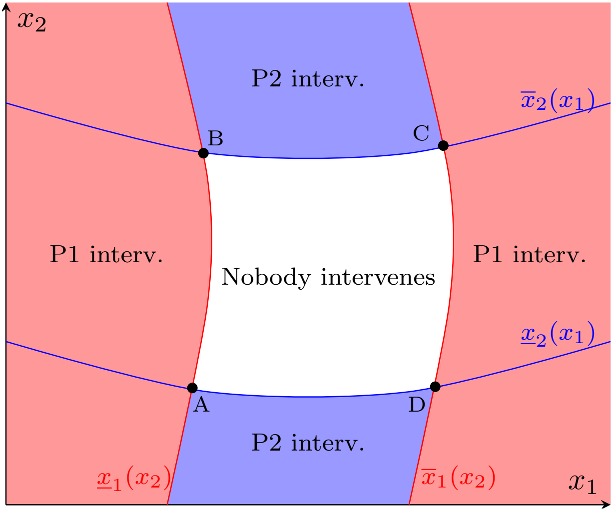
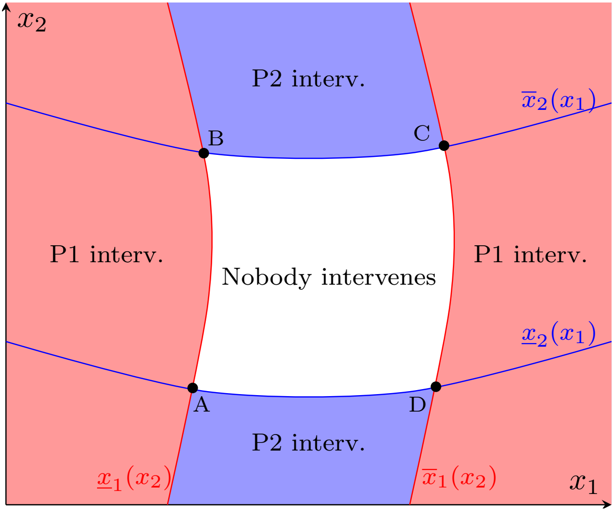
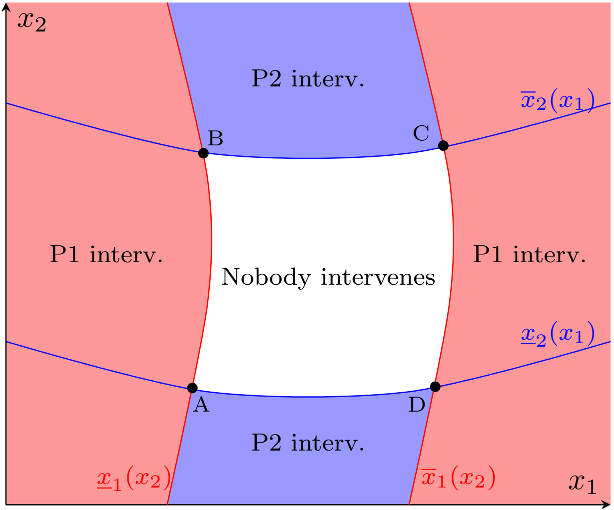
Finally, the region where no one intervenes is

$$\begin{aligned}\{no\ one\ int.\} &= \big(\{P1\ interv.\} \cup \{P2\ interv.\}\big)^c \\ &= \big\{(x_1, x_2) : x_1 \in ]\underline{x}_1(x_2), \bar{x}_1(x_2)[,\ x_2 \in ]\underline{x}_2(x_1), \bar{x}_2(x_1)[\big\}.\end{aligned}$$

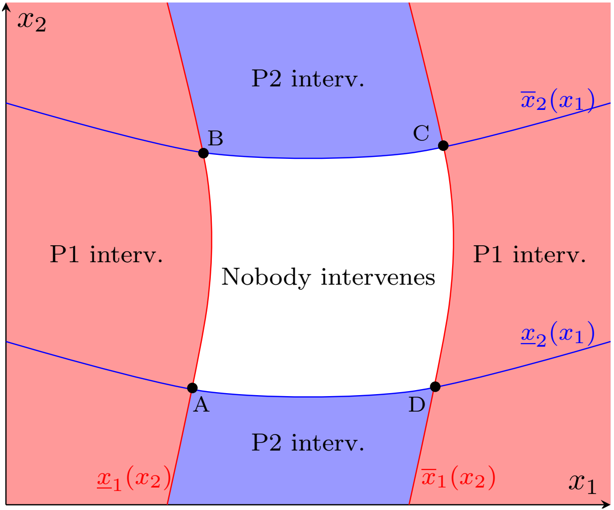

$x_2$
P2 interv.
$\overline{x}_2(x_1)$
B
C
P1 interv.
Nobody intervenes
P1 interv.
$\underline{x}_2(x_1)$
A
D
P2 interv.
$\underline{x}_1(x_2)$
$\overline{x}_1(x_2)$
$x_1$




$$R = \{P1\ interv.\} = \{(x_1, x_2) : x_1 \notin ]\underline{x}_1(x_2), \bar{x}_1(x_2)[\}$$




$$B = \{P2\ interv.\} = \big\{(x_1, x_2) : x_1 \in ]\underline{x}_1(x_2), \bar{x}_1(x_2)[,\ x_2 \notin ]\underline{x}_2(x_1), \bar{x}_2(x_1)[\big\}$$




$$W = \{\textit{no one int.}\} = \left\{(x_1, x_2) : x_1 \in ]\underline{x}_1(x_2), \bar{x}_1(x_2)[,\ x_2 \in ]\underline{x}_2(x_1), \bar{x}_2(x_1)[\right\}$$

Let us now face the QVI problem. The equations read

$$\begin{aligned} &\mathcal{H}_i V_i - V_i = 0, && \text{in } \{\mathcal{M}_j V_j - V_j = 0\}, \\ &\max\big\{\mathcal{A}V_i - \rho V_i + f_i, \mathcal{M}_i V_i - V_i\big\} = 0, && \text{in } \{\mathcal{M}_j V_j - V_j < 0\}. \end{aligned}$$

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We can rewrite them as (where $\varphi_i$ is a sol. to $\mathcal{A}V_i - \rho V_i + f_i = 0$):

$$V_i = \begin{cases} \mathcal{M}_i V_i, & \text{in } \{\mathcal{M}_i V_i - V_i = 0\}, \\ \varphi_i, & \text{in } \{\mathcal{M}_i V_i - V_i < 0, \mathcal{M}_j V_j - V_j < 0\}, \\ \mathcal{H}_i V_i, & \text{in } \{\mathcal{M}_j V_j - V_j = 0\}. \end{cases}$$

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By the practical interpretation of the regions, we get

$$V_1 = \begin{cases} \mathcal{M}_1 V_1, & \text{in } \{\textit{P1 interv.}\}, \\ \varphi_1, & \text{in } \{\textit{no one int.}\}, \\ \mathcal{H}_1 V_i, & \text{in } \{\textit{P2 interv.}\}, \end{cases} V_2 = \begin{cases} \mathcal{H}_2 V_2, & \text{in } \{\textit{P1 interv.}\}, \\ \varphi_2, & \text{in } \{\textit{no one int.}\}, \\ \mathcal{M}_2 V_2, & \text{in } \{\textit{P2 interv.}\}. \end{cases}$$

Up to now, we simply re-wrote the equations (generic argument). Recall:

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We need to estimate: the regions, the functions $\varphi_i$, the operators.

- *The three regions*. Already done!

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We need to estimate: the regions, the functions $\varphi_i$, the operators.

- *The three regions.* $\checkmark$
- *The functions $\varphi_i$.* By definition, $\varphi_i$ is a solution to

$$-\mu(\partial_{x_1} + \partial_{x_2})\varphi_i + \frac{1}{2}\sigma^2(\partial_{x_1} + \partial_{x_2})^2\varphi_i - \rho\varphi_i + f_i = 0.$$

  Idea: change of variable $y_1 = x_1 + x_2$ and $y_2 = x_1 - x_2$, so that the PDE becomes an easily solvable second-order linear ODE.

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We need to estimate: the regions, the functions $\varphi_i$, the operators.

- *The three regions.* ✓
- *The functions $\varphi_i$.* ✓
- *The operators $\mathcal{M}_i, \mathcal{H}_i$.* Heuristic estimates show that

  $$\mathcal{M}_1 V_1(x_1, x_2) = \varphi_1\big(x_1^*(x_2), x_2\big) - c_1, \quad \mathcal{H}_1 V_1(x_1, x_2) = \varphi_1\big(x_1, x_2^*(x_1)\big),$$
  $$\mathcal{M}_2 V_2(x_1, x_2) = \varphi_2\big(x_1, x_2^*(x_1)\big) - c_2, \quad \mathcal{H}_2 V_2(x_1, x_2) = \varphi_2\big(x_1^*(x_2), x_2\big),$$

  where $x_1^*(x_2)$ is a local maximum of $\varphi_1(\cdot, x_2)$ and $x_2^*(x_1)$ is a local maximum of $\varphi_2(x_1, \cdot)$.

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- *The three regions.* $\checkmark$
- *The functions* $\varphi_i$*.* $\checkmark$
- *The operators* $\mathcal{M}_i, \mathcal{H}_i$*.* $\checkmark$

Finally, this leads to the following (class of) candidates.

$$\tilde{V}_1(x_1, x_2) = \begin{cases} \varphi_1\big(x_1^*(x_2), x_2\big) - c_1, & \text{in } R, \\ \varphi_1\big(x_1, x_2\big), & \text{in } W, \\ \varphi_1\big(x_1, x_2^*(x_1)\big), & \text{in } B, \end{cases}$$

$$\tilde{V}_2(x_1, x_2) = \begin{cases} \varphi_2\big(x_1^*(x_2), x_2\big), & \text{in } R, \\ \varphi_2\big(x_1, x_2\big), & \text{in } W, \\ \varphi_2\big(x_1, x_2^*(x_1)\big) - c_2, & \text{in } B. \end{cases}$$

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$$R = \{\textit{P1 interv.}\} = \big\{(x_1,x_2) : x_1 \notin ]\underline{x}_1(x_2),\bar{x}_1(x_2)[\big\}$$
$$B = \{\textit{P2 interv.}\} = \big\{(x_1,x_2) : x_1 \in ]\underline{x}_1(x_2),\bar{x}_1(x_2)[,\ x_2 \notin ]\underline{x}_2(x_1),\bar{x}_2(x_1)[\big\}$$
$$W = \{\textit{no one int.}\} = \big\{(x_1,x_2) : x_1 \in ]\underline{x}_1(x_2),\bar{x}_1(x_2)[,\ x_2 \in ]\underline{x}_2(x_1),\bar{x}_2(x_1)[\big\}$$

$x_1^*(x_2)$ is a local max of $\varphi_1(\cdot,x_2)$ and $x_2^*(x_1)$ is a local max of $\varphi_2(x_1,\cdot)$

an explicit formula for $\varphi_1$ is available

$$\tilde{V}_1(x_1,x_2)=\begin{cases}\varphi_1\big(x_1^*(x_2),x_2\big)-c_1, & \text{in } R,\\ \varphi_1\big(x_1,x_2\big), & \text{in } W,\\ \varphi_1\big(x_1,x_2^*(x_1)\big), & \text{in } B,\end{cases}$$

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$$W=\{no\ one\ int.\}=\big\{(x_1,x_2): x_1\in]\underline{x}_1(x_2),\bar{x}_1(x_2)[,\ x_2\in]\underline{x}_2(x_1),\bar{x}_2(x_1)[\big\}$$

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(Notice: some parameters/function still to be determined!)

**Step 2: conditions on the coefficients.** We now list the conditions that $\tilde{V}_1, \tilde{V}_2$ have to satisfy (basically, this translates into equations on the coefficients). We focus on $\tilde{V}_1$, symmetric arguments for $\tilde{V}_2$.

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First, recall that $x_1^*(x_2)$ is a local maximum of $\varphi_1(\cdot, x_2)$. This corresponds to the f.o.c. $\left(\frac{\partial \varphi_1}{\partial x_1}\right)\big(x_1^*(x_2), x_2\big) = 0$, for each $x_2 \in \mathbb{R}$.

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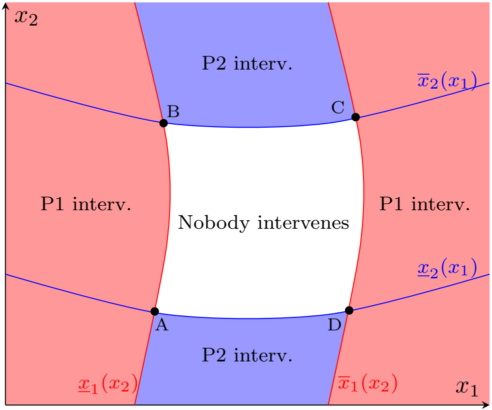
Moreover, recall from the verification theorem that we need

$$\tilde{V}_1 \in C^2(D_2 \setminus \partial D_1) \cap C^1(D_2) \cap C(\mathbb{R}^2),$$

where $D_i = \{\mathcal{M}_i \tilde{V}_i - \tilde{V}_i < 0\}$. As $\tilde{V}_1$ is piecewise defined and each part is $C^\infty$, we need to set some $C^0$-pasting and $C^1$-pasting conditions. In detail, this corresponds to 10 equations.




We set a $C^0$-pasting condition in: the two vertical lines, AD, BC.
We set a $C^1$-pasting condition in: AB, DC.

To sum up, we need to solve the following system.

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$$
\begin{cases}
\left(\frac{\partial \varphi_1}{\partial x_1}\right)\left(x_1^*(x_2), x_2\right) = 0, & x_2 \in \mathbb{R}, \\
\varphi_1\left(x_1^*(x_2), x_2\right) = \varphi_1\left(\underline{x}_1(x_2), x_2\right) + c_1, & x_2 \in \left[x_2^A, x_2^B\right], \\
\varphi_1\left(x_1^*(x_2), x_2\right) = \varphi_1\Big(\underline{x}_1(x_2), x_2^*\left(\underline{x}_1(x_2)\right)\Big) + c_1, & x_2 \in \mathbb{R} \setminus \left[x_2^A, x_2^B\right], \\
\varphi_1\left(x_1^*(x_2), x_2\right) = \varphi_1\left(\bar{x}_1(x_2), x_2\right) + c_1, & x_2 \in \left[x_2^D, x_2^C\right], \\
\varphi_1\left(x_1^*(x_2), x_2\right) = \varphi_1\Big(\bar{x}_1(x_2), x_2^*\left(\bar{x}_1(x_2)\right)\Big) + c_1, & x_2 \in \mathbb{R} \setminus \left[x_2^D, x_2^C\right], \\
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\varphi_1\left(x_1, x_2^*(x_1)\right) = \varphi_1\left(x_1, \bar{x}_2(x_1)\right), & x_1 \in \left]x_1^B, x_1^C\right[, \\
\left(\frac{\partial \varphi_1}{\partial x_1}\right)\left(\underline{x}_1(x_2), x_2\right) = 0, & x_2 \in \left[x_2^A, x_2^B\right], \\
\left(\frac{\partial \varphi_1}{\partial x_2}\right)\left(\underline{x}_1(x_2), x_2\right) = \left(\frac{\partial \varphi_1}{\partial x_2}\right)\left(x_1^*(x_2), x_2\right), & x_2 \in \left[x_2^A, x_2^B\right], \\
\left(\frac{\partial \varphi_1}{\partial x_1}\right)\left(\bar{x}_1(x_2), x_2\right) = 0, & x_2 \in \left[x_2^D, x_2^C\right], \\
\left(\frac{\partial \varphi_1}{\partial x_2}\right)\left(\bar{x}_1(x_2), x_2\right) = \left(\frac{\partial \varphi_1}{\partial x_2}\right)\left(x_1^*(x_2), x_2\right), & x_2 \in \left[x_2^D, x_2^C\right].
\end{cases}
$$

In short, if one finds a solution to the 11+11 equations, the candidate built above satisfies all the assumptions of the verification theorem and we can characterize the value function and the Nash equilibria.

Solution to the 11+11 equations: work in progress...

# A simple example: complete solution

**Appendix: a simple example.** Let us consider the following one-dimensional nonzero-sum impulsive game:

$$J^1(x;\varphi_1,\varphi_2) = \mathbb{E}_x\bigg[\int_0^\infty e^{-\rho s}(X_s - s_1)^3 ds - \sum_{k\geq 1} e^{-\rho\tau_{1,k}} c_1 + \sum_{k\geq 1} e^{-\rho\tau_{2,k}} c_2\bigg],$$

$$J^2(x;\varphi_1,\varphi_2) = \mathbb{E}_x\bigg[\int_0^\infty e^{-\rho s}(s_2 - X_s)^3 ds - \sum_{k\geq 1} e^{-\rho\tau_{2,k}} c_1 + \sum_{k\geq 1} e^{-\rho\tau_{1,k}} c_2\bigg].$$

where $s_1 < s_2$ and, in case of no interventions, we have $dX_s = \sigma dW_s$.

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where $s_1 < s_2$ and, in case of no interventions, we have $dX_s=\sigma dW_s$.

Possible economic interpretation as follows. Let $X$ be the exchange rate between two currencies. The corresponding countries have different targets: player 1 needs a high value, player 2 needs a low rate. Both the players can intervene and move the rate.

We now use the verification theorem, with the following procedure.

- Step 1: we solve the QVI problem to get a pair of (parametric) candidates $\tilde{V}_1, \tilde{V}_2$ for the value functions $V_1, V_2$.
- Step 2: we impose the regularity conditions required in the verification theorem to such candidates.

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- Step 1: we solve the QVI problem to get a pair of (parametric) candidates $\tilde{V}_1, \tilde{V}_2$ for the value functions $V_1, V_2$.
- Step 2: we impose the regularity conditions required in the verification theorem to such candidates.

*Step 1: building a candidate*. As player 1 needs a high rate, we assume his intervention region to be in the form $]-\infty,\bar{x}_1]$. Similarly, we expect the intervention region of player 2 to be in the form $[\bar{x}_2, +\infty[$. The real line is, heuristically, divided into three intervals:

$$]-\infty, \bar{x}_1] = \{\mathcal{M}_1 V_1 - V_1 = 0\}, \text{ where player 1 intervenes,}$$
$$]\bar{x}_1, \bar{x}_2[ = \{\mathcal{M}_1 V_1 - V_1 < 0,\ \mathcal{M}_2 V_2 - V_2 < 0\}, \text{ where no one intervenes,}$$
$$[\bar{x}_2, +\infty[ = \{\mathcal{M}_2 V_2 - V_2 = 0\}, \text{ where player 2 intervenes.}$$

The equations in the QVI problem here read

$$\begin{aligned} &\mathcal{H}_i V_i - V_i = 0, && \text{in } \{\mathcal{M}_j V_j - V_j = 0\}, \\ &\max\left\{\mathcal{A}V_i - \rho V_i + f_i, \mathcal{M}_i V_i - V_i\right\} = 0, && \text{in } \{\mathcal{M}_j V_j - V_j < 0\}, \end{aligned}$$

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which we can rewrite as (where $\varphi_i$ is a sol. to $\mathcal{A}V_i - \rho V_i + f_i = 0$):

$$V_i = \begin{cases} \mathcal{M}_i V_i, & \text{in } \{\mathcal{M}_i V_i - V_i = 0\}, \\ \varphi_i, & \text{in } \{\mathcal{M}_i V_i - V_i < 0, \mathcal{M}_j V_j - V_j < 0\}, \\ \mathcal{H}_i V_i, & \text{in } \{\mathcal{M}_j V_j - V_j = 0\}. \end{cases}$$

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Finally, by the previous partition of the real line, we get

$$V_1 = \begin{cases} \mathcal{M}_1 V_1, & \text{in } ]-\infty, \bar{x}_1], \\ \varphi_1, & \text{in } ]\bar{x}_1, \bar{x}_2[, \\ \mathcal{H}_1 V_1, & \text{in } [\bar{x}_2, +\infty[, \end{cases} \qquad V_2 = \begin{cases} \mathcal{M}_2 V_2, & \text{in } [\bar{x}_2, +\infty[, \\ \varphi_2, & \text{in } ]\bar{x}_1, \bar{x}_2[, \\ \mathcal{H}_2 V_2, & \text{in } ]-\infty, \bar{x}_1]. \end{cases}$$

By heuristic arguments we can estimate $\mathcal{M}_i V_i$ and $\mathcal{H}_i V_i$. This leads to the following (class of) candidates, where $x_i^*$ is a local maximum of $\varphi_i$ in the interval $]\bar{x}_1, \bar{x}_2[$:

$$\widetilde{V}_1(x) = \begin{cases} \varphi_1(x_1^*) - c_1, & \text{if } x \in \,]-\infty, \bar{x}_1], \\ \varphi_1(x), & \text{if } x \in \,]\bar{x}_1, \bar{x}_2[, \\ \varphi_1(x_2^*) + c_2, & \text{if } x \in [\bar{x}_2, +\infty[, \end{cases}$$

$$\widetilde{V}_2(x) = \begin{cases} \varphi_2(x_1^*) + c_2, & \text{if } x \in \,]-\infty, \bar{x}_1], \\ \varphi_2(x), & \text{if } x \in \,]\bar{x}_1, \bar{x}_2[, \\ \varphi_2(x_2^*) - c_1, & \text{if } x \in [\bar{x}_2, +\infty[. \end{cases}$$

By heuristic arguments we can estimate $\mathcal{M}_i V_i$ and $\mathcal{H}_i V_i$. This leads to the following (class of) candidates, where $x_i^*$ is a local maximum of $\varphi_i$ in the interval $]\bar{x}_1, \bar{x}_2[$:

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Notice that some free parameters are present at the moment. We now set such parameters by imposing the regularity conditions.

*Step 2: conditions on the coefficients.* Recall from the verification theorem that we need

$$\widetilde{V}_i \in C^2(D_j \setminus \partial D_i) \cap C^1(D_j) \cap C(S),$$

where $D_i = \{\mathcal{M}_i \widetilde{V}_i - \widetilde{V}_i < 0\}$. In our case, it writes

$$\widetilde{V}_1 \in C^2\Big( ]-\infty, \bar{x}_1[ \cup ]\bar{x}_1, \bar{x}_2[ \Big) \cap C^1\big( ]-\infty, \bar{x}_2[ \big) \cap C(\mathbb{R}),$$
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By definition, we know that

$$\widetilde{V}_1, \widetilde{V}_2 \in C^\infty\Big( \,]-\infty, \bar{x}_1[\, \cup \,]\bar{x}_1, \bar{x}_2[\, \cup \,]\bar{x}_2, +\infty[\, \Big).$$

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Hence, we just have to set six conditions:

- As for $\widetilde{V}_1$: $C^0$-pasting in $\bar{x}_1, \bar{x}_2$ and $C^1$-pasting in $\bar{x}_1$.
- As for $\widetilde{V}_2$: $C^0$-pasting in $\bar{x}_1, \bar{x}_2$ and $C^1$-pasting in $\bar{x}_2$.

To sum up, we have to solve the following system of equations:

$$\begin{cases} \varphi_1'(x_1^*) = 0 \quad \text{and} \quad \varphi_1''(x_1^*) \le 0, & \textit{(optimality of } x_1^*\textit{)} \\ \varphi_1'(\bar{x}_1) = 0, & \textit{(}C^1\textit{-pasting in } \bar{x}_1\textit{)} \\ \varphi_1(\bar{x}_1) = \varphi_1(x_1^*) - c_1, & \textit{(}C^0\textit{-pasting in } \bar{x}_1\textit{)} \\ \varphi_1(\bar{x}_2) = \varphi_1(x_2^*) + c_2, & \textit{(}C^0\textit{-pasting in } \bar{x}_2\textit{)} \end{cases}$$

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with $\bar{x}_1 < x_i^* < \bar{x}_2$, for $i \in \{1, 2\}$. As $\varphi_1, \varphi_2$ are solutions to linear second-order ODEs, we have $\varphi_1 = \varphi_1^{A_{11}, A_{12}}$ and $\varphi_2 = \varphi_2^{A_{21}, A_{22}}$, with $A_{ij} \in \mathbb{R}$.

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*Conclusion.* In short, provided that a solution to such system exists, we have two well-defined candidates $\widetilde{V}_1, \widetilde{V}_2$ for the value functions $V_1, V_2$. We can now apply the verification theorem.

**Proposition**

Under some minor assumptions, assume that the eight parameters $\bar{x}_i, x_i^*, A_{ij}$ solve the system above. Then, the value functions are

$$V_1(x) = \begin{cases} \varphi_1(x_1^*) - c_1, & \text{if } x \in\, ]-\infty, \bar{x}_1], \\ \varphi_1(x), & \text{if } x \in\, ]\bar{x}_1, \bar{x}_2[, \\ \varphi_1(x_2^*) + c_2, & \text{if } x \in [\bar{x}_2, +\infty[, \end{cases}$$

$$V_2(x) = \begin{cases} \varphi_2(x_1^*) + c_2, & \text{if } x \in\, ]-\infty, \bar{x}_1], \\ \varphi_2(x), & \text{if } x \in\, ]\bar{x}_1, \bar{x}_2[, \\ \varphi_2(x_2^*) - c_1, & \text{if } x \in [\bar{x}_2, +\infty[. \end{cases}$$

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Moreover, the Nash equilibria are characterized as follows.

- Player $i$ intervenes if and only if the process hits $\bar{x}_i$.
- When intervening, player $i$ shifts the process to the state $x_i^*$.

# Competition in retail markets:
# a simpler one-player model

**One player: the model.** We consider an energy retailer who buys energy in the wholesale market and re-sells it to final consumers, in each $t \in [0, \infty[$.

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- The price management policy is determined by the sequence $u = \{(\tau_k, \delta_k)\}_k$ (impulse control), where $\tau_k$ are the intervention times and $\delta_k$ are the corresponding shifts.
- Intervening has a (fixed) cost, denoted $c$. The retailer also faces operational costs, quadratic w.r.t. his market share $\Phi_t$.
- The player's market share depends on $X_t = P_t - S_t$: in our model, $\Phi_t = \Phi(X_t) = \min\left\{1,\ \max\{0, -1/\Delta(X_t - \Delta)\}\right\}$.

The retailer buys/re-sells energy ($\rightarrow$ continuous-time revenue),

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The retailer buys/re-sells energy ($\rightarrow$ continuous-time revenue), pays quadratic operational costs ($\rightarrow$ continuous-time spending),

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This is a standard stochastic control problem with impulse controls. Our goal is to characterize the value function and the optimal price management policy.

**One player: characterizing $V$.** If $V : \mathbb{R} \to \mathbb{R}$

- is a solution to $\max\left\{\mathcal{A}V - \rho V - f, \mathcal{M}V - V\right\} = 0$,
- is bounded and in $C^2(\mathbb{R} \setminus \{\mathcal{M}V - V < 0\}) \cap C^1(\mathbb{R})$,

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The procedure is as follows.

- First, we get a candidate by solving the PDE above (the candidate depends on some parameters) .
- Then, we impose the regularity conditions (this corresponds to algebraic equations on the parameters).

Finally, we get the following result.

## Proposition

The value function is

$$V(x) = \begin{cases} \varphi_{A_1,A_2}(x), & \text{in } ]\underline{x}, \bar{x}[, \\ \varphi_{A_1,A_2}(x^*) - c, & \text{in } \mathbb{R} \backslash ]\underline{x}, \bar{x}[, \end{cases}$$

where $\varphi_{A_1,A_2}$ is an explicit function and the five parameters $(A_1, A_2, \underline{x}, \bar{x}, x^*)$ are the unique solution to a suitable algebraic system of equations. Moreover, the optimal control is as follows:

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Here, the interaction between opposing retailers is not directly modelled, but only implicitly considered (the player's market share decreases as his income rises) $\longrightarrow$ we introduce a second player.