Optimal liquidation in an Almgren-Chriss type model with Lévy processes and finite time horizons

Junwei Xu
joint work with Arne Løkka

Department of Mathematics
London School of Economics and Political Science

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What is an optimal liquidation problem?

We consider an investor who aims to sell a large amount of shares. Since the trading volume by this investor is large, due to a lack of liquidity, the market price will drop during selling, potentially resulting in huge execution costs. The investor seeks to find an optimal strategy, maximising the final cash subject to some optimisation criterion.

The Almgen-Chriss framework

For $t \geq 0$, let $Y_t \geq 0$ be the position in a stock and $\xi_t \in \mathbb{R}$ be the execution speed. So with $Y_0 = y$,

$$Y_t = y + \int_0^t \xi_s \, ds.$$

The observed market price of this stock at time t is

$$S_t = \underbrace{S_t^u}_{\text{unaffected price}} + \underbrace{\alpha(Y_t - Y_0)}_{\text{permanent impact}} + \underbrace{F(\xi_t)}_{\text{temporary impact}},$$

where $\alpha \geq 0$ is the coefficient of permanent impact and $F : \mathbb{R} \to \mathbb{R}$ is the temporary impact function which is increasing and satisfies $F(0) = 0$.

Our model: Lévy unaffected price

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a complete filtered probability space satisfying the usual conditions, which supports a one dimensional, non-trivial, $\mathbb{F}$-adapted Lévy process L. Our unaffected price is given by

$$\begin{aligned} S_t^u &= s + L_t \\ &= s + \mu t + \sigma W_t + \int_{\mathbb{R}} x \, \tilde{N}(t, dx). \end{aligned}$$

Our model: Admissible strategies

For any time horizon $T \in (0, \infty)$ and any initial stock position $y \geq 0$, any admissible liquidation strategy $Y \in \mathcal{A}(T, y)$ satisfies

- Y is adapted and absolutely continuous;
- $Y_0 = y$, $Y_T = 0$ and $Y_t \geq 0$ for $t \in [0, T]$.

$\mathcal{A}_D(T, y)$ is the set of all deterministic admissible strategies.

Our model: General temporary impact function

The temporary impact function $F : \mathbb{R} \to \mathbb{R}$ satisfies

- $F \in C(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{0\})$;
- $F(0) = 0$;
- the function $x \mapsto xF(x)$ is strictly convex on $\mathbb{R}$;
- some other technical conditions

Cash position

Stock price at time $t \geq 0$ is given by

$$S_t = s + L_t + \alpha(Y_t - Y_0) + F(\xi_t).$$

Let C be the process of cash position with $C_0 = c$. Then for any $Y \in \mathcal{A}(T, y)$, the total cash at time T is given by

$$\begin{aligned} C_T &= c - \int_0^T S_t \, dY_t \\ &= c + sy - \frac{1}{2}\alpha y^2 + \int_0^T Y_{t-} \, dL_t - \int_0^T \xi_t F(\xi_t) \, dt \qquad a.s. \end{aligned}$$

The large investor's optimisation problem

Suppose the investor has a constant absolute risk aversion, and wants to maximise the expected utility of final cash. Therefore, his problem is given by

$$\sup_{Y \in \mathcal{A}(T,y)} \mathbb{E}\big[-\exp(-AC_T)\big]$$

where $A > 0$ is the risk aversion. This problem is equivalent to

$$\inf_{Y \in \mathcal{A}(T,y)} \mathbb{E}\left[\exp\left(-\int_0^T AY_{t-}\,dL_t + A\int_0^T \xi_t F(\xi_t)\,dt\right)\right].$$

Problem simplification

- Define the function $\kappa_A : [0, \bar{\delta}_A) \to \mathbb{R}$ by

$$\kappa_A(x) = \kappa(-Ax),$$

where $A > 0$ is a constant, κ is the cumulant generating function of L_1.

- Therefore,

$$M_t = \exp\left(\int_0^t -AY_{u-}\, dL_u - \int_0^t \kappa_A(Y_u)\, du \right), \qquad t \in [0, T]$$

is a martingale.

- If we assume that $y \in [0, \bar{\delta}_A)$, then based on the idea in **Schied, Schöneborn and Tehranchi (2010)**, we calculate that

$$
\begin{aligned}
&\inf_{Y \in \mathcal{A}(T,y)} \mathbb{E}\left[\exp\left(-\int_0^T AY_{t-}\,dL_t + A\int_0^T \xi_t F(\xi_t)\,dt\right)\right] \\
&= \inf_{Y \in \mathcal{A}(T,y)} \mathbb{E}\left[\exp\left(-\int_0^T AY_{t-}\,dL_t - \int_0^T \kappa_A(Y_t)dt\right)\times \right. \\
&\qquad \left. \times \exp\left(\int_0^T \kappa_A(Y_t) + A\xi_t F(\xi_t)\,dt\right)\right] \\
&= \inf_{Y \in \mathcal{A}(T,y)} \widetilde{\mathbb{E}}\left[\exp\left(\int_0^T \kappa_A(Y_t) + A\xi_t F(\xi_t)\,dt\right)\right] \\
&= \inf_{Y \in \mathcal{A}_D(T,y)} \widetilde{\mathbb{E}}\left[\exp\left(\int_0^T \kappa_A(Y_t) + A\xi_t F(\xi_t)\,dt\right)\right] \\
&= \inf_{Y \in \mathcal{A}_D(T,y)} \exp\left[\int_0^T \left(\kappa_A(Y_t) + A\xi_t F(\xi_t)\right)dt\right] \\
&\sim \inf_{Y \in \mathcal{A}_D(T,y)} \int_0^T \left(\kappa_A(Y_t) + A\xi_t F(\xi_t)\right)dt,,
\end{aligned}
$$

Simplified problem:

$$\inf_{Y \in \mathcal{A}_D(T,y)} \int_0^T \Big(\kappa_A(Y_t) + A\xi_t F(\xi_t) \Big)\, dt,$$

where $(T, y) \in (0, \infty) \times [0, \bar{\delta}_A)$ and

$$dY_t = \xi_t \, dt.$$

- The Euler-Lagrange equation:

$$\kappa_A'(Y_t) - \frac{d}{dt}[AF(\xi_t) + A\xi_t F'(\xi_t)] = 0$$

  with $Y_0 = y$ and $Y_T = 0$. But F' is not differentiable!

- The Beltrami identity:

$$\kappa_A(Y_t) - A\xi_t^2 F'(\xi_t) \equiv \text{const}$$

  with $Y_0 = y$ and $Y_T = 0$. It is equivalent to the Euler-Lagrange equation if F' is differentiable.

A general optimisation problem

- $\phi : \mathbb{R} \times \mathbb{R} \to \overline{\mathbb{R}}$ is proper and lower semi-continuous.
- $\mathcal{D} := \mathcal{D}_1 \times \mathcal{D}_2 := \{(x, \dot{x}) \,|\, \phi(x, \dot{x}) < \infty\}$ has non-empty interior.
- $\phi \in C^1(\text{int}(\mathcal{D}))$.
- For any $x \in \mathcal{D}_1$, $\phi(x, \cdot)$ is strictly convex on $\mathcal{D}_2$.

Consider the problem

$$\inf_{x(\cdot)} \int_a^b \phi\big(x(t), \dot{x}(t)\big)\, dt, \tag{1}$$

where any admissible $x(\cdot)$ satisfies that

- $x(\cdot)$ is absolutely continuous;
- $x(a) = x_a$ and $x(b) = x_b$ with x_a, $x_b \in \mathcal{D}_1$;
- $\int_a^b \big|\phi\big(x(t), \dot{x}(t)\big)\big|\, dt < \infty$.

Necessary and sufficient conditions for the optimiser

For any $x, \dot{x} \in \mathbb{R}$ and $p \in \overline{\mathbb{R}}$, write

$$H(x, \dot{x}, p) = \dot{x}p - \phi(x, \dot{x}),$$

and define the Hamiltonian to be

$$\mathcal{H}(x, p) = \sup_{\dot{x} \in \mathbb{R}} H(x, \dot{x}, p).$$

Theorem 1 (necessary)

Let $\hat{x}(\cdot)$ be an optimal admissible function for (1) and $\dot{\hat{x}}(\cdot)$ be the derivative of $\hat{x}(\cdot)$. Suppose $\hat{x}(\cdot)$ is Lipschitz continuous with associated Lipschitz constant belonging to the interior of $\mathcal{D}_2$. We also suppose that $\hat{x}(\cdot)$ takes values in the interior of $\mathcal{D}_1$. Then there exists a function $p : [a, b] \to \mathbb{R}$ satisfying

$$\dot{p}(t) = -H_x\big(\hat{x}(t), \dot{\hat{x}}(t), p(t)\big), \quad a.e.\ t \in [a, b],$$

and

$$H\big(\hat{x}(t), \dot{\hat{x}}(t), p(t)\big) = \max_{\dot{x}\in\mathbb{R}} H\big(\hat{x}(t), \dot{x}(t), p(t)\big), \quad a.e.\ t \in [a, b].$$

If in addition that

$$\mathcal{H}_x\big(\hat{x}(t), p(t)\big) = H_x\big(\hat{x}(t), \dot{\hat{x}}(t), p(t)\big), \quad a.e.\ t \in [a, b],$$

then the optimal function $\hat{x}(\cdot)$ satisfies

$$\phi\big(\hat{x}(t), \dot{\hat{x}}(t)\big) - \dot{\hat{x}}(t)\phi_{\dot{x}}\big(\hat{x}(t), \dot{\hat{x}}(t)\big) \equiv K, \quad a.e.\ t \in [a, b],$$

where K is some constant.

Theorem 2 (sufficient)

Suppose for any $x \in \mathcal{D}_1$ *and any* $p \in \{\phi_{\dot{x}}(x, \dot{x}) \,|\, \dot{x} \in \mathcal{D}_2\}$, $\mathcal{H}(\cdot, p)$ *is concave. Let* $\hat{x}(\cdot)$ *be an admissible function such that* $t \mapsto \phi_{\dot{x}}\big(\hat{x}(t), \dot{\hat{x}}(t)\big)$ *is absolutely continuous on* $[a, b]$*. Suppose*

$$\phi\big(\hat{x}(t), \dot{\hat{x}}(t)\big) - \dot{\hat{x}}(t)\phi_{\dot{x}}\big(\hat{x}(t), \dot{\hat{x}}(t)\big) \equiv K, \quad \text{for all } t \in [a, b];$$

and when $\dot{\hat{x}}(\cdot) = 0$ *a.e. on some interval contained in* $[a, b]$*, we have* $\mathcal{H}_x\big(\hat{x}(\cdot), p(\cdot)\big) = 0$ *on the same interval. Then such* $\hat{x}(\cdot)$ *is optimal for problem (1).*

The optimal liquidation strategy

- Recall the simplified optimal liquidation problem: $(T, y) \in (0, \infty) \times [0, \overline{\delta}_A)$,

$$\inf_{Y \in \mathcal{A}_D(T,y)} \int_0^T \Big(\kappa_A(Y_t) + A\xi_t F(\xi_t) \Big)\, dt.$$

- Consider the Beltrami identity

$$K^{T,y} + \kappa_A(Y_t) = A\xi_t^2 F'(\xi_t).$$

Define the continuous functions $G^- : [0, \infty) \to (-\infty, 0]$ and $G^+ : [0, \infty) \to [0, \infty)$ to be the inverses of $x \mapsto x^2 F'(x)$ restricted on intervals $(-\infty, 0]$ and $[0, \infty)$, respectively.

- Given any $(T, y) \in (0, \infty) \times [0, \bar{\delta}_A)$, the **unique** optimal strategy satisfies either

$$\frac{d\hat{Y}_t^{T,y}}{dt} = \hat{\xi}_t^{T,y} = G^-\left(\frac{K^{T,y} + \kappa_A\big(\hat{Y}_t^{T,y}\big)}{A}\right)$$

or

$$\frac{d\hat{Y}_t^{T,y}}{dt} = \hat{\xi}_t^{T,y} = G^+\left(\frac{K^{T,y} + \kappa_A\big(\hat{Y}_t^{T,y}\big)}{A}\right)\mathbb{1}_{[0,\tau^y)}(t) + G^-\left(\frac{K^{T,y} + \kappa_A\big(\hat{Y}_t^{T,y}\big)}{A}\right)\mathbb{1}_{[\tau^y,T]}(t),$$

where $\tau^y = \inf\{t \geq 0 \,|\, \hat{Y}_t^{T,y} = \kappa_A^{-1}(-K^{T,y})\}$.

- $K^{T,y}$ is uniquely determined by T and y.

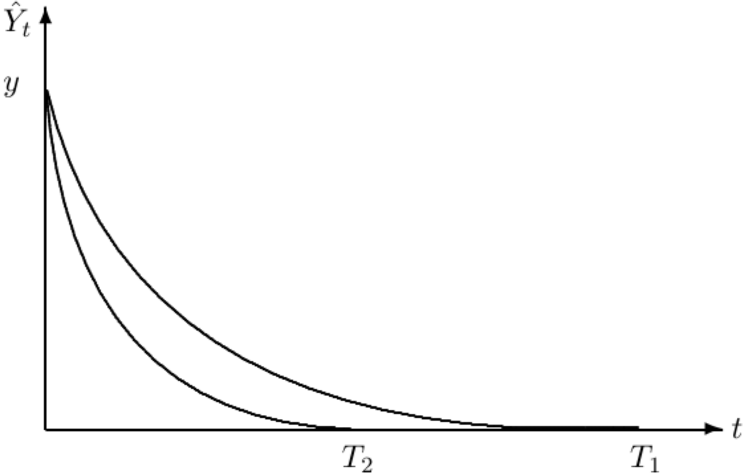

Figure: An illustration of optimal liquidation trajectories for $\mu \leq 0$.

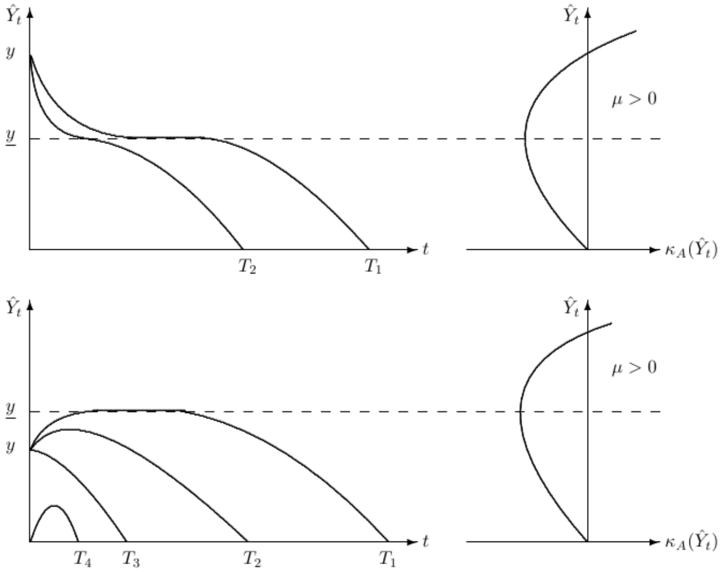

Figure: An illustration of optimal liquidation trajectories for $\mu > 0$.

Existence of price manipulation

- Price manipulation (**Huberman and Stanzl 2004**): there exits a round-trip strategy which gives out strictly positive proceeds in average.
- Suppose $\mu > 0$. Let $c = y = 0$ and Y^0 be the liquidation strategy of doing nothing. Then

$$-\exp(-Ac) = \mathbb{E}\Big[-\exp\big(-AC_T^{Y^0}\big)\Big] < \mathbb{E}\Big[-\exp\big(-AC_T^{\hat{Y}}\big)\Big] \leq -\exp\Big(-A\mathbb{E}\big[C_T^{\hat{Y}}\big]\Big).$$

Therefore,

$$0 = c < \mathbb{E}\big[C_T^{\hat{Y}}\big].$$

The value funtion

- When $\mu \le 0$, for $(T, y) \in (0, \infty) \times [0, \bar{\delta}_A)$,

$$V(T,y) = \int_0^y \frac{-\kappa_A(u)}{G^-\left(\frac{K^{T,y}+\kappa_A(u)}{A}\right)} - AF\left(G^-\left(\frac{K^{T,y}+\kappa_A(u)}{A}\right)\right) du.$$

- For $\mu > 0$, if $(T, y) \in (0, \infty) \times [\underline{y}, \bar{\delta}_A)$ or $(T, y) \in (0, \widetilde{\tau}^y] \times [0, \underline{y})$,

$$V(T,y) = \int_0^y \frac{-\kappa_A(u)}{G^-\left(\frac{K^{T,y}+\kappa_A(u)}{A}\right)} - AF\left(G^-\left(\frac{K^{T,y}+\kappa_A(u)}{A}\right)\right) du$$
$$+ (T - \widetilde{T} - \widetilde{T}^y_-)\kappa_A(\underline{y})\, \mathbb{1}_{(\widetilde{T}^y,\infty)\times[\underline{y},\bar{\delta}_A)}(T,y);$$

and if $(T, y) \in (\widetilde{\tau}^y, \infty) \times [0, \underline{y})$, then

$$V(T,y) = \int_{\kappa_A^{-1}(-K^{T,y})}^y \frac{-\kappa_A(u)}{G^+\left(\frac{K^{T,y}+\kappa_A(u)}{A}\right)} - AF\left(G^+\left(\frac{K^{T,y}+\kappa_A(u)}{A}\right)\right) du$$
$$+ \int_0^{\kappa_A^{-1}(-K^{T,y})} \frac{-\kappa_A(u)}{G^-\left(\frac{K^{T,y}+\kappa_A(u)}{A}\right)} - AF\left(G^-\left(\frac{K^{T,y}+\kappa_A(u)}{A}\right)\right) du$$
$$+ (T - \widetilde{T} - \widetilde{T}^y_+)\kappa_A(\underline{y})\, \mathbb{1}_{(\widetilde{T}^y_+ + \widetilde{T}\,,\,\infty)}(T).$$

Connection to the infinite time horizon model

- If $\mu > 0$, $\lim_{T\to\infty} \hat{Y}^{T,y}$ is a strategy which never goes to 0, and $\lim_{T\to\infty} V(T,y) = \infty$.

- For $\mu \leq 0$, the optimal speed and the value function in the infinite time horizon model are given by (our another working paper)

$$\hat{\xi}_t^{\infty,y} = G^{-}\left(\frac{\kappa_A\big(\hat{Y}_t^{\infty,y}\big)}{A}\right), \quad t \geq 0,$$

and

$$V^{\infty}(y) = \int_0^y \frac{-\kappa_A(u)}{G^{-}\left(\frac{\kappa_A(u)}{A}\right)} - AF\left(G^{-}\left(\frac{\kappa_A(u)}{A}\right)\right) du, \quad y \in [0, \bar{\delta}_A).$$

We have that as $T \to \infty$,

$$\hat{\xi}_t^{T,y} \to \hat{\xi}_t^{\infty,y},$$
$$\hat{Y}_t^{T,y} \to \hat{Y}_t^{\infty,y},$$
$$V(T,y) \to V^{\infty}(y).$$

Write $\left(C_t^{\hat{Y}^{T,y}}\right)_{t\geq 0} = \left(C_{t\wedge T}^{\hat{Y}^{T,y}}\right)_{t\geq 0}$, then as $T \to \infty$,

$$C_{\infty}^{\hat{Y}^{T,y}} \xrightarrow{L^2(\mathbb{P})} C_{\infty}^{\hat{Y}^{\infty,y}}.$$

Thanks for your attention!