Mean reflected SDE

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Joint work with P. Briand, A. Guillin and C. Labart

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Motivations : measure at risk

- In finance :
  - The *risk measure* of a position is the (minimal) amount of own found needed by a company to hold the position
  - Given a risk measure, we can define a set of *acceptable positions* : the set of positions that do not require any own found to be hold
  - Given a set of *acceptable positions* for a company, we can define a risk measure : for any position the risk measure is the minimal amount of cash that makes the position acceptable

- Mathematical modelization :
  - $(\Omega, \mathcal{F})$
  - $X \ : \ \Omega \ni \omega \mapsto X(\omega) \in \mathbb{R}$ value of the position
  - $\mathcal{A} \subset \mathbb{L}_2(\Omega)$ set of acceptable positions
  - $\rho_{\mathcal{A}} = \inf\{m \in \mathbb{R} \ : \ m + X \in \mathcal{A}\}$ risk measure associated to the acceptable positions set
    - increasing
    - "translating invariant" : $\rho(X+m) = \rho(X) + m, \ m \in \mathbb{R}$

Motivations : context

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- Example : the "Value at Risk" at level α :

$$\mathrm{VAR}_\alpha(\mathrm{X}) = \inf\{m \in \mathbb{R} \ : \ \mathbb{P}(m + X) \leqslant \alpha\}$$

i.e.

$$h : x \mapsto \mathbf{1}_{x \geqslant 0} - (1 - \alpha), \quad 0 < \alpha < 1$$

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- Reflected stochastic differential equation → but the reflection acts on the law

↳ Mean reflected stochastic differential equation

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$\hookrightarrow$ The process K is deterministic

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Definition A solution of the MR-SDE is a couple (X,K) satisfying the above system with K a non-decreasing deterministic function satisfying $K_0=0$.

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  - Fixed point : coefficients b ans σ Lipschitz
    - Initialization $X^0 = Y^0 = x_0$
    - Set $K_t^0 = \sup_{s\leqslant t} \inf\{x \geqslant 0 : \mathbb{E}h(x + Y_s^0) \geqslant 0\} = \sup_{s\leqslant t} G_0^+(\mu_s^0)$
    - We obtain $X^1 = Y^0 + K^0$
    - *and so on..*

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$$\begin{aligned} &\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t + \mathrm{d}K_t, \quad X_0 = x_0 \\ &\forall t \in [0, T] : \ \mathbb{E}h(X_t) \geqslant 0 \quad \int_0^t \mathbb{E}[h(X_s)]\mathrm{d}K_s = 0 \end{aligned}$$

where B is a Brownian motion define on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$

- Auxiliary dynamic : $\mathrm{d}Y_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t$
  - Fixed point : coefficients b ans σ Lipschitz
    - Initialization $X^0 = Y^0 = x_0$
    - Set $K_t^0 = \sup_{s\leqslant t} \inf\{x \geqslant 0 : \mathbb{E}h(x + Y_s^0) \geqslant 0\} = \sup_{s\leqslant t} G_0^+(\mu_s^0)$
    - We obtain $X^1 = Y^0 + K^0$
    - *and so on..*
  - Standard computations give :

$$\mathbb{E} \sup_{t\leqslant T} |X_t^{n+1} - X_t^n|^2 \leqslant C_T \sup_{t\leqslant T} |K_t^{n+1} - K_t^n|^2$$

  - S.C. for convergence ?

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    ↳ S.C. for G_0^+ to be Lipschitz : h bi-Lipschitz

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- **Theorem :** If b and σ are Lipschitz continuous and if in addition h is a bi-Lipschitz function then the MR-SDE as a unique solution

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- **Corollary :** If in addition h is a C^2 function, the Stietljes measure $\mathrm{d}K$ is absolutely continuous w.r.t. the Lebesgue measure with density :

$$k_t = \frac{(\mathbb{E}\mathcal{L}h(X_t))^-}{\mathbb{E}h'X_t}\mathbf{1}_{\mathbb{E}h(X_t)=0}.$$

Mean reflected SDE and interacting reflected particles system

$$
\begin{cases}
\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t + \mathrm{d}\sup_{s\leqslant t} G_0^+(\mu_s), \quad \mu_s = \mathcal{L}(Y_s) \\
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- Mean reflection $\leftrightarrow$ non linear reflection (McKean-Vlasov sense)
  ↳ Interacting reflected particle system with oblique reflection ?

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Result : The rate of convergence is of order $N^{-1/9}$ under our standing assumptions and of order $N^{-1/2}$ if h is C^2.

- The rate of convergence relies on $\mathbb{E}\sup_{t\leqslant T}|G_0^+(\mu_t) - G_0^+(\bar{\mu}_t^N)|$ where $\bar{X}_t^i = \int_0^t b(X_s^i)\mathrm{d}s + \int_0^t \sigma(X_s^i)\mathrm{d}B_s^i + \sup_{s\leqslant t} G_0^+(\bar{\mu}_s^N), \quad \bar{\mu}_s^N = N^{-1}\sum_{i=1}^N \delta_{\bar{X}_s^i}$

Mean reflected SDE numerical algorithm

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- Take advantage of the propagation of chaos phenomenon

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- Euler discretization of the particle system :

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$$\begin{cases} X^i_{t+h} = X^i_t + hb(X^i_t) + \sigma(X^i_t)(B^i_{t+h} - B^i_t) + K^N_{t+h} - K^N_t, \\ K^N_{t+h} - K^N_t = \inf\{x \geqslant 0 : N^{-1}\sum_{i=1}^{N} h(x + Y^i_{t+h}) \geqslant 0\} \\ Y^i_{t+h} = X^i_t + hb(X^i_t) + \sigma(X^i_t)(B^i_{t+h} - B^i_t) \end{cases}$$

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Result :

- $|\mathrm{Error}| \leqslant C\left(n^{-1/2} + N^{-1/2}\right)$, if h is C^2
- $|\mathrm{Error}| \leqslant C\left(n^{-1/2} + N^{-1/9}\right)$, otherwise

MR-SDE : numerical illustrations deterministic drift

- $T > 0$, $h : \mathbb{R} \ni x \mapsto x - p \in \mathbb{R}$ and $\mathrm{d}X_t = -\mu \mathrm{d}t + \sigma \mathrm{d}B_t + \mathrm{d}K_t$

$$\hookrightarrow \quad K_t = (p + \mu t - x_0)^+$$

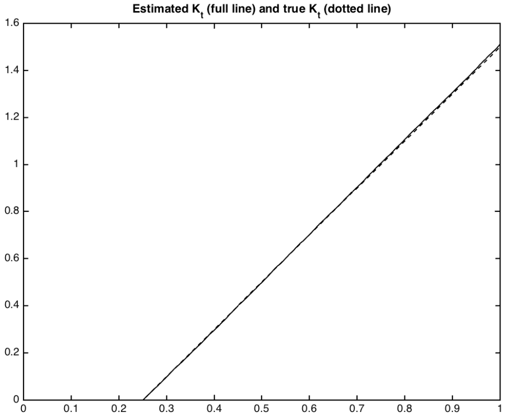

FIGURE: Parameters : $n = 500$, $N = 10000$, $T = 1$, $\mu = 2$, $\sigma = 1$, $x_0 = 1$, $p = 1/2$

- $T > 0$, $h : \mathbb{R} \ni x \mapsto x - p \in \mathbb{R}$ and $\mathrm{d}X_t = -(\mu - \epsilon B_t)\mathrm{d}t + \sigma \mathrm{d}B_t + \mathrm{d}K_t$

$$\hookrightarrow K_t = \left(p - x_0 + \mu t - \sigma\epsilon\frac{t^2}{2}\right) 1_{[0,\bar{t})}(t) + \left(p - x_0 + \mu\bar{t} - \sigma\epsilon\frac{\bar{t}^2}{2}\right) 1_{[\bar{t},t^\star]}(t) + o(\epsilon)$$

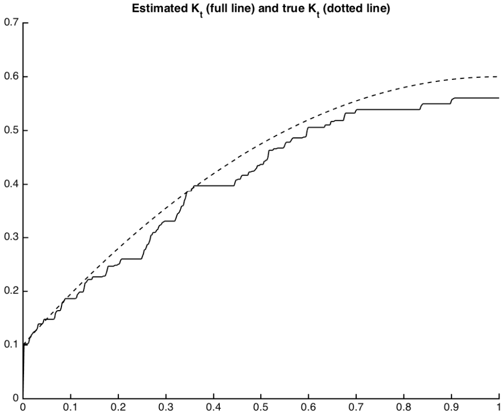

FIGURE: Parameters : $n = 500$, $N = 10000$, $T = 1$, $\mu = 1$, $\epsilon = 1/10$, $\sigma = 1/\epsilon$, $x_0 = 1$, $p = 1.1$

- $T > 0$, $h : \mathbb{R} \ni x \mapsto x - p \in \mathbb{R}$ and $\mathrm{d}X_t = -(\mu + aX_t)\mathrm{d}t + \sigma \mathrm{d}B_t + \mathrm{d}K_t$

$$\hookrightarrow \quad K_t = (ap - \mu)(t - t^\star)1_{t \geqslant t^\star}, \text{ where } t^\star = \frac{1}{a}\left(\ln(x_0 + \mu/a) - \ln(p + \mu/a)\right)$$

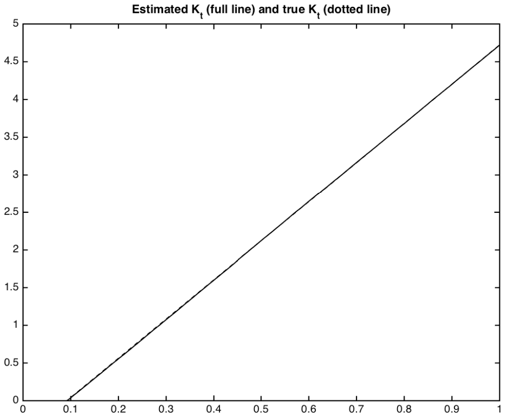

FIGURE: Parameters : $n = 500,\ N = 10000,\ T = 1,\ \mu = 2.1,\ a = 1,\ \sigma = 1,\ x_0 = 1,\ p = 3.6$

- $T > 0$, $h : x \mapsto x + \alpha \sin(x) - p$, $-1 < \alpha < 1$ and $\mathrm{d}X_t = -(\mu + aX_t)\mathrm{d}t + \sigma \mathrm{d}B_t + \mathrm{d}K_t$

$$\hookrightarrow \mathrm{d}K_t = e^{-at}\mathrm{d}\sup_{s\leqslant t}\left(F_s^{-1}(0)\right)^+$$

$$F_t : x \quad \mapsto \quad \left\{ e^{-at}\left(x_0 - \mu\left(\frac{e^{at}-1}{a}\right) + x\right) + \alpha \exp\left(-e^{-at}\frac{\sigma^2}{a}\sinh(at)\right) \right.$$
$$\left. \times \sin\left(e^{-at}\left(x_0 - \mu\left(\frac{e^{at}-1}{a}\right) + x\right)\right) - p \right\}.$$

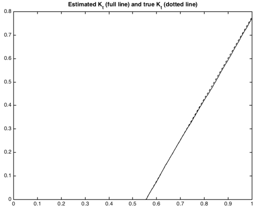

FIGURE: Parameters :
$n = 1000, \; N = 10000, \; T = 1, \; \mu = 1, \; \sigma = 1, \; p = 1.1, \; \alpha = 1/2, \; x_0 = \alpha + p + 1/2$

MR-SDE : illustrations non-linear constraint

- $T > 0$, $h : x \mapsto x + \alpha \sin(x) - p$, $-1 < \alpha < 1$ and $\mathrm{d}X_t = -(\mu + aX_t)\mathrm{d}t + \sigma \mathrm{d}B_t + \mathrm{d}K_t$

$$\hookrightarrow \mathrm{d}K_t = e^{-at}\mathrm{d}\sup_{s \leqslant t}\left(F_s^{-1}(0)\right)^+$$

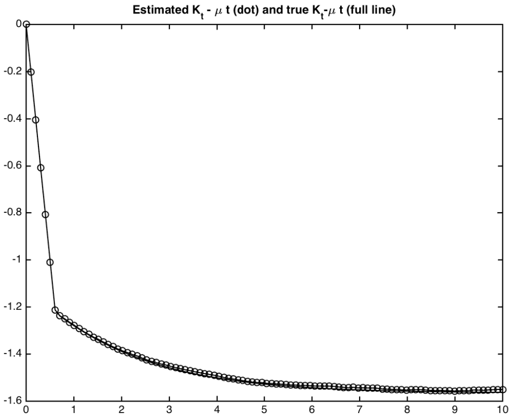

FIGURE: Parameters :
$n = 100$, $N = 10000$, $T = 2$, $\beta = 2$, $\sigma = 1$, $p = 3\pi/2$, $\alpha = 1/2$, $x_0 = 2 * \pi$

Thanks !