Reflected stochastic differential equations driven by G-Brownian motion

SOUMANA HIMA Abdoulaye

UNIVERSITE DE RENNES 1

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Plan

- G-expectation
- Reflected G-Brownian motion
- Reflected G-SDEs

Sublinear expectation

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  - Positive homogeneity: $\widetilde{\mathbb{E}}[\lambda X] = \lambda \widetilde{\mathbb{E}}[X], \ \ \forall \lambda \geq 0, \ X \in \mathcal{H}$.

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  - Positive homogeneity: $\widetilde{\mathbb{E}}[\lambda X] = \lambda \widetilde{\mathbb{E}}[X], \ \forall \lambda \geq 0, \ X \in \mathcal{H}$.
- $X = (X_1, \ldots, X_d)$ is G-normally distributed if $\forall a, b \geq 0$, we have:
$$aX + b\overline{X} \stackrel{d}{=} \sqrt{a^2 + b^2} X,$$
where $\bar{X}$ in an independent copy of X and G denotes the function
$$G(A) := \frac{1}{2} \mathbb{E}[(AX, X)] : \mathbb{S}_d \to \mathbb{R}.$$

G-expectation

Let $\Omega_T = C_0\left([0, T], \mathbb{R}^d\right)$, $B_t(\omega) = \omega_t$ and

$$\mathcal{H}_T^0 := \left\{\varphi\left(B_{t_1}, \ldots, B_{t_n}\right) : n \geq 1;\ \varphi \in C_{b,Lip}\left(\mathbb{R}^{d\times n}\right)\right\}.$$

- G-expectation is a sublinear expectation defined by

$$\widehat{\mathbb{E}}[X] = \widetilde{\mathbb{E}}\left[\varphi\left(\sqrt{t_1 - t_0}\xi_1, \cdots, \sqrt{t_m - t_{m-1}}\xi_m\right)\right],$$

for $X = \varphi\left(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_m} - B_{t_{m-1}}\right)$, where $\xi_1, \cdots, \xi_n$ are identically distributed G-normal distribution in a sublinear expectation space $\left(\widetilde{\Omega}, \widetilde{\mathcal{H}}, \widetilde{\mathbb{E}}\right)$.

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- The conditional G-expectation $\widehat{\mathbb{E}}_t$ of X knowing $\mathcal{H}_t^0$, is defined by

$$\widehat{\mathbb{E}}_{t_i}[X] = \tilde{\varphi}\left(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_i} - B_{t_{i-1}}\right),$$

  where

$$\tilde{\varphi}\left(x_1, \cdots, x_i\right) = \widehat{\mathbb{E}}\left[\varphi\left(x_1, \cdots, x_i, B_{t_{i+1}} - B_{t_i}, \cdots, B_{t_m} - B_{t_{m-1}}\right)\right].$$

G-capacity

- [Denis, Hu, Peng] There exists a tight family $\mathcal{P}$ of probability measures on $(\Omega_T, \mathcal{B}(\Omega_T))$, such that

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- The Choquet capacity is defined by:

$$\hat{c}(A) := \sup_{P\in\mathcal{P}} P(A), \quad A \in \mathcal{B}(\Omega_T).$$

A set $A \subset \mathcal{B}(\Omega_T)$ is called polar if $\hat{c}(A) = 0$.

Set of processes

- $$M_{b,0}(0,T) = \left\{ \eta : \eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) \mathbf{1}_{[t_j, t_{j+1})}(t),\ \forall N \in \mathbb{N}, \right.$$
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$$\left. 0 = t_0 < \cdots < t_N = T,\ \xi_j \in B_b(\Omega_{t_j}) \right\}.$$
- For $p \geq 1$, $M_*^p(0,T)$ denote the completion of $M_{b,0}(0,T)$ under
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- For $\eta \in M_{b,0}(0,T)$, the stochastic integral with respect to B is defined by
$$I(\eta) = \int_0^T \eta_t dB_t^a := \sum_{j=0}^{N-1} \xi_j(\omega)(B_{t_{j+1}}^a - B_{t_j}^a).$$

The Domain D

- For a domain $D \subset \mathbb{R}^d$, $r > 0$ and $x \in \partial D$, denote by

$$\mathcal{N}_{x,r} = \left\{ \mathbf{n} \in \mathbb{R}^d, \ |\mathbf{n}| = 1 \ \text{ and } \ B(x - r\mathbf{n}, r) \cap D = \emptyset \right\}$$
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  - **CONDITION (A).** uniform exterior sphere condition

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  - **CONDITION (B).** $\exists \delta > 0$ and $\beta \in [1, \infty)$ such that $\forall x \in \partial D$ there exists a unit vector l_x such that

$$\langle l_x, \mathbf{n} \rangle \geq \frac{1}{\beta} \quad \text{for any} \quad \mathbf{n} \in \bigcup_{y \in B(x,\delta) \cap \partial D} \mathcal{N}_y.$$

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Under the conditions (A) and (B), we have:

- $$\mathbf{n} \in \mathcal{N}_x \Leftrightarrow \langle y - x, \mathbf{n} \rangle + \frac{1}{2r_0} |y - x|^2 \geq 0, \quad \forall y \in \overline{D}.$$

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  - ∇U is bounded and Lipschitz continous.

Skorohod problem

For $w \in C\left([0,T], \mathbb{R}^d\right)$, $w(0) = 0$ and $x_0 \in \overline{D}$, we consider the following Skorohod's equation:

$$\xi(t) = x_0 + w(t) + \phi(t), \quad t \in [0,T]. \tag{1}$$

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$$\phi(t) = \int_0^t \mathbf{n}(s) d\left|\phi\right|_s, \quad \text{and} \quad \left|\phi\right|_t = \int_0^t 1_{\partial D}(\xi(s)) d\left|\phi\right|_s,$$

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  where $\mathbf{n}(s) \in \mathcal{N}_{\xi(s)}$ if $\xi(s) \in \partial D$.
- [Lions et Sznitman 1984] under the conditions (A) and (B), if w is is continuous, there exists a unique solution (ξ, ϕ) for the Skorokhod problem.

Reflected G-Brownian motion

Consider :

$$X_t = x_0 + B_t + K_t, \quad 0 \le t \le T. \tag{2}$$

We call (X, K) a solution of G-Skorohod problem (2) if there exists a polar set A such that:

- $X, \; K \in M_*^2\left([0, T], \mathbb{R}^d\right)$ q.s. continuous

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- $$\forall \omega \in A^c, \; X_t(\omega) \in \overline{D}, \; \left|K_\cdot(\omega)\right|_0^T < \infty \text{ and } K_0(\omega) = 0;$$
- $\forall \omega \in A^c$,
$$\begin{aligned} |K|_t(\omega) &= \int_0^t \mathbf{1}_{\{X_s(\omega) \in \partial D\}} d|K|_s(\omega), \\ K_t(\omega) &= \int_0^t \mathbf{n}(X_s(\omega)) \, d|K|_s(\omega), \; \forall t \in [0, T]. \end{aligned}$$

Penalization

[Saisho, Tanaka 1987] For any $m \geq 1$, consider the equation

$$X_t^m = x_0 + B_t - \frac{m}{2} \int_0^t \nabla U(X_s^m) ds, \quad 0 \leq t \leq T \qquad \text{q.s.,} \qquad (3)$$

$$K_t^m = -\frac{m}{2} \int_0^t \nabla U(X_s^m) ds.$$

- The G-SDE (3) (cf [Lin 2013]) has a unique solution $X^m \in M_*^p\left([0,T], \mathbb{R}^d\right)$ and $K^m \in M_*^p\left([0,T], \mathbb{R}^d\right), \ p \geq 1$.

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- [Saisho, Tanaka 1987]) Under the conditions (A) and (B), there exists a polar set A such that $\forall \omega \in A^c$, then $\left(X_\cdot^m(\omega), K_\cdot^m(\omega)\right)_{m \geq 1}$ converges uniformly to the solution of Skorohod problem

$$X_t(\omega) = x_0 + B_t(\omega) + K_t(\omega).$$

Penalization

- For $\omega \in A^c$ and $\alpha \in \left]0, 1/2\right[$, the modulus of uniform continuity of $\omega.$ is defined for $h \in [0, T]$ by

$$\Delta^{\alpha}_{0,T,h}(\omega) := \sup_{\substack{0 \le s < t \le T \\ |t-s| \le h}} |\omega_t - \omega_s| \le h^{\alpha} \sup_{\substack{0 \le s < t \le T \\ |t-s| \le h}} \frac{|\omega_t - \omega_s|}{|t-s|^{\alpha}} \le h^{\alpha} \|\omega\|_{\alpha},$$

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where

$$\|\omega\|_{\alpha} = \sup_{0 \le s < t \le T} \frac{|\omega_t - \omega_s|}{|t-s|^{\alpha}}.$$

- For $m \in \mathbb{N}^*$, set:

$$\varepsilon^{\alpha}_m(\omega) = \frac{12 e^L}{m^{\alpha}} \|\omega\|_{\alpha},$$

where $2L$ is the Lipschitz constant of ∇U.

Estimates of K

Let $0 < \varepsilon_m^{\alpha}(\omega) < \min(\delta/2, r_0/2) := \tau$, and set :

$$
\begin{aligned}
T_{m,0} &= \inf\{t \geq 0 : \overline{X_t^m(\omega)} \in \partial D\}, \\
t_{m,n} &= \inf\left\{t > T_{m,n-1} : \left|\overline{X_t^m(\omega)} - \overline{X_{T_{m,n-1}}^m(\omega)}\right| \geq \delta/2\right\}, \qquad (4) \\
T_{m,n} &= \inf\{t \geq t_{m,n} : \overline{X_t^m(\omega)} \in \partial D\}.
\end{aligned}
$$

Lemma

If $T_{m,n-1} \leq s < t \leq T_{m,n}$, *then*

$$\Delta_{s,t}(X_\cdot^m(\omega)) \leq 9\beta\left(\Delta_{s,t}(B_\cdot(\omega)) + \varepsilon_m^{\alpha}(\omega)\right)\exp\{\gamma(\|B_\cdot(\omega)\|_T + \delta)\}, \qquad (5)$$

where $\gamma = \dfrac{2\kappa^2\beta}{r_0}$, $\|B_\cdot(\omega)\|_T = \sup\{|B_t(\omega)| : 0 \leq t \leq T\}$ *and*
$\Delta_{s,t}(X_\cdot(\omega)) = \sup\{|X_{t_2}(\omega) - X_{t_1}(\omega)| : s \leq t_1 < t_2 \leq t\}$.

Estimates of K

Proposition

If D satisfies (A) and (B), then for all $\omega \in A_m$, we have

$$|K^m_\cdot|_T(\omega) \leq C \|B_\cdot(\omega)\|_\alpha^{1+1/\alpha} \exp\left\{\gamma\left(1+\frac{1}{\alpha}\right)\|B_\cdot(\omega)\|_T\right\}, \tag{6}$$

where $C := (r_0, \beta, \delta, T, L)$.

Estimates of K

- Since $\varepsilon_m^{\alpha}(\omega) < r_0/2$, we have

$$\nabla U\left(X_t^m(\omega)\right) = 2\left(X_t^m(\omega) - \overline{X_t^m(\omega)}\right), \qquad 0 \le t \le T.$$

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$$\nabla U(X_t^m(\omega)) = 2\left(X_t^m(\omega) - \overline{X_t^m(\omega)}\right), \qquad 0 \leq t \leq T.$$

- From condition (B), we gets for $T_{m,n-1} \leq s < t \leq T_{m,n}$

$$|K_\cdot^m|_t^s(\omega) \leq \beta\left\{\Delta_{s,t}(X_\cdot^m(\omega)) + \Delta_{s,t}(B_\cdot(\omega))\right\}.$$

Estimates of K

- Since $\varepsilon_m^\alpha(\omega) < r_0/2$, we have

$$\nabla U\left(X_t^m(\omega)\right) = 2\left(X_t^m(\omega) - \overline{X_t^m(\omega)}\right), \qquad 0 \le t \le T.$$

- From condition (B), we gets for $T_{m,n-1} \le s < t \le T_{m,n}$

$$\left|K_\cdot^m\right|_t^s(\omega) \le \beta\left\{\Delta_{s,t}\left(X_\cdot^m(\omega)\right) + \Delta_{s,t}\left(B_\cdot(\omega)\right)\right\}.$$

- And (5) implies

$$\left|K_\cdot^m\right|_t^s(\omega) \le 10\beta^2\lambda\{\Delta_{s,t}\left(B_\cdot(\omega)\right) + \varepsilon_m^\alpha(\omega)\}, \qquad (7)$$

where $\lambda = \exp\left\{\gamma\left(\|B_\cdot(\omega)\|_T + \delta\right)\right\}$.

Estimates of K

- (4) and (5) imply

$$\Delta_m := \frac{\delta - 4\varepsilon_m^{\alpha}(\omega)}{20\beta^2\lambda} - \varepsilon_m^{\alpha}(\omega) \leq \Delta_{T_{m,n-1},t_{m,n}}(B_{.}(\omega)),$$

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- since $\lim_{n\to\infty} \Delta_m = \frac{\delta}{20\beta^2\lambda} > 0$, $\exists m_0 \geq 1$ such that $\forall m \geq m_0$,

$$\frac{\delta}{25\beta^2\lambda} \leq \Delta_{T_{m,n-1},t_{m,n}}(B_{.}(\omega)).$$

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$$\frac{\delta}{25\beta^2\lambda} \leq \Delta_{T_{m,n-1},t_{m,n}}(B_{.}(\omega)).$$

- For $h = \left(\frac{\delta}{25\beta^2\lambda\|B_{.}(\omega)\|_{\alpha}}\right)^{1/\alpha} > 0$, $\forall m \geq m_0$, $n > \dfrac{T}{h} \Rightarrow T_{m,n} > T$.

$$(7) \Rightarrow |K^m|_T(\omega) \leq 10(\frac{T}{h}+1)\beta^2\lambda\{\Delta_{0,T}(B_{.}(\omega)) + \varepsilon_m^{\alpha}(\omega)\}.$$

Convergence

Lemma

- *There exist two positive constants C and λ, such that:*

$$\hat{c}\left(A_m^c\right) \leq Ce^{-\lambda m}, \text{ where } A_m = \{\omega \in \Omega_T;\ \varepsilon_m^\alpha(\omega) \leq \tau\}. \tag{8}$$

- $\forall \nu \geq 0$, $\exists C_\nu$ *dependent only on* ν *such that*

$$\mathbb{E}\left[\exp\left(\nu \sup_{0\leq t\leq T} |B_t|\right)\right] \leq C_\nu. \tag{9}$$

Proposition

If the domain D satisfies conditions (A) and (B), then $\forall p > \frac{2\alpha^2}{(1+\alpha)(1-2\alpha)}$:

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |X_t^m|^p\right] + \mathbb{E}\left[(|K^m|_T)^p\right] \leq C_p \text{ where } C_p := (T, p, C_G, L, \beta, \delta) > 0$$

Convergence

- $\forall \omega \in A_m^c$, $|K^m|_T(\omega) \leq \frac{m}{2} T\Lambda$. Letting $\mu = p\left(1+\frac{1}{\alpha}\right)$, we have

$$\mathbb{E}\left[(|K^m|_T)^p\right] \leq C_p \left\{ \left\{ \mathbb{E}\left[\left(\sup_{0\leq s<t\leq T} \frac{|B_t - B_s|}{|t-s|^\alpha}\right)^{\mu/\alpha}\right]\right\}^\alpha \times \left\{\mathbb{E}\left[\exp\left(\frac{4\gamma\beta\mu}{1-\alpha} \sup_{0\leq t\leq T} |B_t|\right)\right]\right\}^{1-\alpha} + m^p e^{-\lambda m^\alpha} \right\}$$

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- On the other hand, from

$$X_t^m = x_0 + B_t + K_t^m,$$

we get

$$\sup_{0\le t\le T} |X_t^m|^p \le C_p \left(|x_0|^p + \sup_{0\le t\le T} |B_t|^p + (|K^m|_T)^p \right).$$

Convergence

Theorem

Suppose that the domain D satisfies conditions (A) and (B). Then, there exists a unique pair of processes $(X, K) \in \left(M_*^p([0,T])\right)^2$, *satisfying q.s.:*

- $$X_t = x_0 + B_t + K_t, \quad 0 \le t \le T, \ q.s.$$
- X *takes values in* $\overline{D}$,
- $K_0 = 0$, K *is bounded variation and*

$$|K|_t = \int_0^t \mathbf{1}_{\partial\mathcal{D}}(X_s)\, d|K|_s, \qquad K_t = \int_0^t \mathbf{n}(X_s)\, d|K|_s.$$

Convergence

- Since the sequence $(X^m)_{m\geq 1}$ converges q.s. to X, for $p \geq 1$ we have

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |X_t|^p\right] \leq \mathbb{E}\left[\liminf_{m\to\infty}\left(\sup_{0\leq t\leq T} |X_t^m|^p\right)\right] \leq C_p.$$

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- For $\varepsilon > 0$, set

$$B_\varepsilon = \bigcup_{k\geq 1} \bigcup_{n\geq m_k} \left\{\omega \in \Omega;\ \sup_{0\leq t\leq T} |X^m(\omega) - X(\omega)| > \frac{1}{k}\right\},$$

we get $\hat{c}(B_\varepsilon) \leq \varepsilon$.

Convergence

- Since the sequence $(X^m)_{m\geq 1}$ converges q.s. to X, for $p \geq 1$ we have

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|X_t|^p\right] \leq \mathbb{E}\left[\liminf_{m\to\infty}\left(\sup_{0\leq t\leq T}|X_t^m|^p\right)\right] \leq C_p.$$

- For $\varepsilon > 0$, set

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we get $\hat{c}(B_\varepsilon) \leq \varepsilon$.
- Let $m_\varepsilon \in \mathbb{N}$ such that

$$\forall m \geq m_\varepsilon \Rightarrow \sup_{0\leq t\leq T}|X^m(\omega) - X(\omega)|\,\mathbf{1}_{B_\varepsilon^c} \leq \varepsilon,$$

then, $\forall m \geq m_\varepsilon$ we have

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|X_t^m - X_t|^p\right] \leq \varepsilon^p + 2^p\varepsilon^{1/2}C_p.$$

Case of G-Itô processes

Consider the G-Itô processes

$$M_t = \int_0^t \alpha_s ds + \int_0^t \eta_s d\langle B\rangle_s + \int_0^t \beta_s dB_s, \qquad 0 \leq t \leq T,$$

where $\alpha,\ \eta,\ \beta \in M_*^p([0,T])$, $p > 2$, such that there exists $L > 0$

$$\sup_{0\leq t\leq T} \mathbb{E}\left[|\alpha_t|^p\right] + \sup_{0\leq t\leq T} \mathbb{E}\left[|\eta_t|^p\right] + \sup_{0\leq t\leq T} \mathbb{E}\left[|\beta_t|^p\right] \leq L.$$

- Then, for $0 < s < t < T$ and $p > 2$, $\exists C_p := (p, C_G, L)$ such that :

$$\mathbb{E}\left[|M_t - M_s|^p\right] \leq C_p \left|t - s\right|^{\frac{p}{2}}.$$

Case of G-Itô processes

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where $\alpha,\ \eta,\ \beta \in M_*^p([0,T])$, $p > 2$, such that there exists $L > 0$

$$\sup_{0\le t\le T} \mathbb{E}[|\alpha_t|^p] + \sup_{0\le t\le T} \mathbb{E}[|\eta_t|^p] + \sup_{0\le t\le T} \mathbb{E}[|\beta_t|^p] \le L.$$

- Then, for $0 < s < t < T$ and $p > 2$, $\exists C_p := (p, C_G, L)$ such that :

$$\mathbb{E}[|M_t - M_s|^p] \le C_p |t-s|^{\frac{p}{2}}.$$

- There exists a polar set A such that $\forall \omega \in A^c$, $M_.(\omega)$ is $\alpha-$Hölder continuous, for any $\alpha \in \left[0, \frac{1}{2} - \frac{1}{p}\right[$.

Case of G-Itô processés

By the same method as for the $G-$Brownian motion, we get

Corollary

Let $x_0 \in \overline{D}$. there exists a unique pair of process $(X, K) \in \left(M_*^p([0,T])\right)^2$, satisfying q.s.

- $$X_t = x_0 + M_t + K_t, \quad 0 \leq t \leq T, \; q.s.$$
- *X takes values in $\overline{D}$,*
- *$K_0 = 0$, K is bounded variation and*

$$K_t = \int_0^t n(s)\, d|K|_s, \quad |K|_t = \int_0^t \mathbf{1}_{\partial\overline{D}}(X_s)\, d|K|_s.$$

Existence and uniqueness of reflected G-SDE

For $x_0 \in \overline{D}$, consider the following equation :

$$X_t = x_0 + \int_0^t g\left(s, X_s\right) dB_s + \int_0^t f\left(s, X_s\right) ds - K_t, \quad 0 \leq t \leq T, \quad (10)$$

where $f : \Omega \times [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $g : \Omega \times [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d \times n}$ satisfy:

- (**H1**) For any $x \in \mathbb{R}^d$ the processes

$$f_i\left(\cdot, x\right), \; g_{ij}\left(\cdot, x\right) \in M_*^2\left([0, T]\right), \; i = 1, \ldots, d \text{ and } j = 1, \ldots, n;$$

- (**H2**) $f_i, \; g_{ij}$ are uniformly bounded and $\forall t \in [0, T], \;\; \forall x, y \in \mathbb{R}^d$,

$$\left|g_{ij}\left(t, x\right) - g_{ij}\left(t, y\right)\right| + \left|f_i\left(t, x\right) - f_i\left(t, y\right)\right| \leq L\left|x - y\right|.$$

Existence and uniqueness of reflected G-SDE

$(X, K) \in M_*^2\left([0, T]; \mathbb{R}^d\right) \times M_*^2\left([0, T]; \mathbb{R}^d\right)$ q.s. continuous on $[0, T]$ is a solution of (10) if:

- (X, K) satisfy (10);

Existence and uniqueness of reflected G-SDE

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- (X,K) satisfy (10);
- There exists a polar set A such that $\forall \omega \in A^c,\ X_t(\omega) \in \overline{D}$, $|K_\cdot|_T^0(\omega) < \infty$ and $K_0(\omega) = 0$;

Existence and uniqueness of reflected G-SDE

$(X, K) \in M_*^2([0, T]; \mathbb{R}^d) \times M_*^2([0, T]; \mathbb{R}^d)$ q.s. continuous on $[0, T]$ is a solution of (10) if:

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- There exists a polar set A such that $\forall \omega \in A^c$, $X_t(\omega) \in \overline{D}$, $|K_\cdot|_T^0(\omega) < \infty$ and $K_0(\omega) = 0$;
- $\forall \omega \in A^c$

$$
\begin{aligned}
|K(\omega)|_t &= \int_0^t \mathbf{1}_{(X_s(\omega) \in \partial \overline{D})} d|K(\omega)|_s, \\
K_t(\omega) &= \int_0^t \xi_s d|K(\omega)|_s,
\end{aligned}
$$

with $\xi_s \in \mathbf{n}(X_s)$.

Existence and uniqueness of reflected G-SDE

Condition (C): [Lions, Sznitman 1984] $\exists \phi \in C_b^2\left(\mathbb{R}^d\right)$ and $\gamma > 0$ such that :

$$\forall x \in \partial D,\ \forall y \in \overline{D},\ \forall \mathbf{n} \in \mathcal{N}_x\ \ \langle y - x, \mathbf{n} \rangle + \frac{1}{\gamma} \langle \nabla \phi(x), \mathbf{n} \rangle |y - x|^2 \geq 0.$$

If (Y, K) is a solution of Skorokhod problem

$$\left(x_0 + \int_0^{\cdot} g(s, X_s)\, dB_s + \int_0^{\cdot} f(s, X_s)\, ds, D, n(\cdot) \right),$$

and $y_t \in \overline{D}$, we have:

$$\frac{1}{\gamma} \int_0^t |Y_s - y_s|^2 \langle \nabla \phi(Y_s), dK_s \rangle \leq \int_0^t \langle Y_s - y_s, dK_s \rangle . \qquad (11)$$

Existence and uniqueness of reflected G-SDE

Proposition

*Assume that D satisfies (**A**), (**B**) and (**C**) and for $i = 1,\ 2,\ g^i,\ f^i$ verify (**H1**) and(**H2**). For $x_0 \in \bar{D}$, let $\left(X^i, K^i\right)$ be the solution of*

$$X_t^i = x_0 + \int_0^t g^i\left(s, X_s^i\right) dB_s + \int_0^t f^i\left(s, X_s^i\right) ds - K_t^i.$$

$\exists C := (d, T, L, M, C_D, C_G) > 0$ *such that :*

$$\begin{aligned} & \mathbb{E}\left[\sup_{0\leq u\leq T}\left|X_u^1 - X_u^2\right|^4\right] + \mathbb{E}\left[\sup_{0\leq t\leq T}\left|K_t^1 - K_t^2\right|^4\right] \\ \leq \quad & C\left(\mathbb{E}\left[\sup_{0\leq u\leq T}\left|\hat{g}_u\right|^4\right] + \mathbb{E}\left[\sup_{0\leq u\leq T}\left|\hat{f}_u\right|^4\right]\right), \end{aligned}$$

where $\hat{g}_s = g^1\left(s, X_s^2\right) - g^2\left(s, X_s^2\right)$ *and* $\hat{f}_s = f^1\left(s, X_s^2\right) - f^2\left(s, X_s^2\right)$.

Existence and uniqueness of reflected G-EDS

Theorem

*Suppose the domain D satisfies the conditions (A), (B) and (C) and functions g_{ij}, f_i verify the hypothesis (**H1**) and (**H2**). Then for every $x_0 \in \overline{D}$ the equation (10) has a unique solution.*

For $i \in \{1, 2\}$, let $X^i \in M_*^2\left([0, T]; \overline{D}\right)$ and

$$Y_t^i = x_0 + \int_0^t g\left(s, X_s^i\right) dB_s + \int_0^t f\left(s, X_s^i\right) ds - K_t^i.$$

It is shown that:

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} \left|Y_t^1 - Y_t^2\right|^4\right] \leq C\mathbb{E}\left[\sup_{0 \leq t \leq T} \left|X_t^1 - X_t^2\right|^4\right].$$