Optimal Bounds for Risk Measures

Nabil Kazi-Tani
Laboratoire de Sciences Actuarielle et Financière (SAF),
Lyon 1 University
Joint work with Stéphane Loisel

Young Researchers Meeting in Probability, Numerics and Finance
Le Mans
June 30, 2016

The Problem

Let $R : L^p \to \mathbb{R}$ be a given functional ($p \geq 1$).

We want to solve the following optimization problem:

$$\sup_{X \in \mathcal{L}} R(X)$$

where $\mathcal{L}$ denotes the set of probability laws on $\mathbb{R}$ such that

$$\mathbb{E}[g_i(X)] = c_i, \ \forall i \in I,$$

where $\{g_i, i \in I\}$ is a finite set of given functions and $\{c_i, i \in I\}$ are given constants.

The Problem

Interesting criteria:

- $R(X) := \rho(X)$ is a given risk measure.
- $R(X) := \mathbb{E}[v(X)]$.

The Problem

Interesting criteria:

- $R(X) := \rho(X)$ is a given risk measure.
- $R(X) := \mathbb{E}[v(X)]$.

Interesting constraints:

- $g_i(x) = x^i$, $i = 0, \ldots, k$.
- The functions $\{g_i, i \in I\}$ form a Tchebycheff system.

Risk Measures

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a given probability space.

We consider a mapping $\rho: \ L^p \to \mathbb{R} \cup \{\infty\}$:

- If $X \geq Y$ $\mathbb{P}$-a.s. then $\rho(X) \geq \rho(Y)$. (Losses orientation)
- $\rho(X+m) = \rho(X) + m, \ m \in \mathbb{R}$. (Cash additivity property: Capital requirement)
- Law invariance : If $X = Y$ in law (under $\mathbb{P}$) then $\rho(X) = \rho(Y)$.

Risk Measures

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a given probability space.

We consider a mapping $\rho: \ L^p \to \mathbb{R} \cup \{\infty\}$:

- If $X \geq Y$ $\mathbb{P}$-a.s. then $\rho(X) \geq \rho(Y)$. (Losses orientation)
- $\rho(X + m) = \rho(X) + m, \ m \in \mathbb{R}$. (Cash additivity property: Capital requirement)
- Law invariance : If $X = Y$ in law (under $\mathbb{P}$) then $\rho(X) = \rho(Y)$.
- If X cannot be used as a hedge for Y (X and Y comonotone variables), then no possible diversification (comonotonic risk measures): $\rho(X + Y) = \rho(X) + \rho(Y)$.

Monetary risk measures

Growing need of regulation professionals and VaR drawbacks conducted to an axiomatic analysis of required solvency capital.

- Artzner, Delbaen, Eber, and Heath (1999) (**Coherent case**)
- Frittelli, M. and Rosazza Gianin, E. (2002) (**Convex case**)
- Föllmer, H. and Schied, A. (2004) (**Monography**)
- Bion-Nadal, (2008-2009); Bion-Nadal and Kervarec (2010), Cheridito, Delbaen, and Kupper (2004) (**Dynamic case**)
- Acciaio (2007, 2009), Barrieu and El Karoui (2008), Jouini, Schachermayer and Touzi (2006,2008), Kervarec (2008) (**Inf-convolution**)

Many other references...

Motivations

- Quantification of model uncertainty: Barrieu and Scandolo, *Assessing financial model risk*, European J. of Operational Research (2015)

Motivations

- Quantification of model uncertainty: Barrieu and Scandolo, *Assessing financial model risk*, European J. of Operational Research (2015)

Proposed metric:

$$RM(X_0, \mathcal{L}) := \frac{\overline{\rho}(\mathcal{L}) - \rho(X_0)}{\overline{\rho}(\mathcal{L}) - \underline{\rho}(\mathcal{L})}$$

where

$$\overline{\rho}(\mathcal{L}) := \sup_{X \in \mathcal{L}} \rho(X) \quad \text{and} \quad \underline{\rho}(\mathcal{L}) := \inf_{X \in \mathcal{L}} \rho(X)$$

Motivations

- Model free pricing in insurance.

Compute

$$\sup_{X \in \mathcal{L}} \mathbb{E}[v(X)]$$

where v is a given convex function.

- Jansen, Haezendonck and Goovaerts (1986)
- Hurlimann (1988)

Tools

- Law invariance : Duality between the Distribution formulation and the Quantile formulation.
- Approximation of quantile and distribution curves by constrained step functions.
- Convex functions : continuity properties.

Objective

Solve the following optimization problem :

$$\sup_{X \in \mathcal{L}} \rho(X)$$

where $\mathcal{L}$ denotes the set of probability laws on $\mathbb{R}$ such that

$$\mathbb{E}[X^i] = c_i, \ \forall i = 1, \dots, k.$$

Methodology

We reformulate the problem in the following manner :

$$\sup_{q \in \mathcal{Q}} \Phi(q)$$

where $\mathcal{Q}$ denotes the set of quantile functions of probability laws on $\mathbb{R}$ with the given moment constraints, and where Φ is such that $\rho(X) = \Phi(q_X)$.

A result

Theorem

Assume that Φ is **linear**, then

$$\sup_{q \in \mathcal{Q}} \Phi(q) = \sup_{q \in \mathcal{Q}_k^*} \Phi(q)$$

where $\mathcal{Q}_k^*$ denotes the set of quantile functions of atomic probability measures on $\mathbb{R}$ with at most $k+1$ atoms, and satisfying the moment constraints.

A result

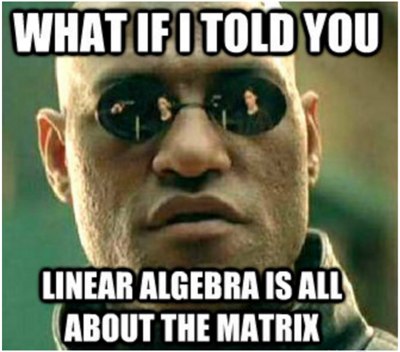
Idea of the proof.

- We first remark that Φ is continuous: Biagini and Fritteli (2009), *On the extension of the Namioka-Klee theorem and on the Fatou property for Risk Measures*.
- We approach every $q \in \mathcal{Q}$ by a $q^* \in \mathcal{Q}^*$ in the L^p norm. ($\mathcal{Q}^*$ denotes the set of quantile functions of atomic measures with a finite number of atoms)
- The two previous points give us:

$$\sup_{q \in \mathcal{Q}} \Phi(q) = \sup_{q \in \mathcal{Q}^*} \Phi(q)$$

A result

- Then to reduce the supremum only over $\mathcal{Q}_k^*$ we follow the explicit contruction given in by Hoeffding, *The extrema of the expected value of a function of independent random variables*, Ann. Math. Statist. (1955).



Application to DRM

Application to the case of distortion risk measures:

A distortion risk measure is **law invariant** and can be written

$$\Phi(\overline{q}) = \int_0^1 \overline{q}(u) d\psi(u)$$

where ψ is a given distortion function. It is a **linear** functional in the $\overline{q}$ variable !

Application to DRM

Assume that $k = 2$. To obtain a superior bound, all one need to compute is:

$$\sup_{\overline{q}\in\overline{\mathcal{Q}}_{m_1,m_2}} \Phi(\overline{q})$$

$$= \sup_{p_i,a_i} \left(\psi(p_1)a_3 + a_2\{\psi(p_1+p_2)-\psi(p_1)\} + a_3\{1-\psi(p_1+p_2)\}\right)$$

under the constraints

$$\begin{pmatrix} 1 & 1 & 1 \\ a_1 & a_2 & a_3 \\ a_1^2 & a_2^2 & a_3^2 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} 1 \\ m_1 \\ m_2 \end{pmatrix}$$

Application to DRM

Completely different approach to compute the former supremum :

Let μ and ν be two arbitrary probability measures on $\mathbb{R}$. We say that μ dominates ν in the *first order stochastic dominance* if

$$\int g d\mu \geq \int g d\nu \text{ for all continuous, bounded and increasing function } g.$$

We say that μ dominates ν in the *second order stochastic dominance* if

$$\int g d\mu \geq \int g d\nu \text{ for all bounded, increasing and concave function } g.$$

Application to DRM

The distorsion risk measures **preserve** the first and second order stochastic dominance.

- Question : Can we find a maximal distribution for the first order stochastic dominance?

Application to DRM

The distorsion risk measures **preserve** the first and second order stochastic dominance.

- Question : Can we find a maximal distribution for the first order stochastic dominance?
- Yes: Results from the 80's summarized in Hurlimann, *Extremal moment methods and stochastic orders: application in actuarial science*, Bol. Asoc. Mat. Venez. (2008).

Application to DRM

The distorsion risk measures **preserve** the first and second order stochastic dominance.

- Question : Can we find a maximal distribution for the first order stochastic dominance?
- Yes: Results from the 80's summarized in Hurlimann, *Extremal moment methods and stochastic orders: application in actuarial science*, Bol. Asoc. Mat. Venez. (2008).

When $k = 2$, $m_1 = 0$ and $m_2 = 1$, the worst case first order stochastic dominance cumulative distribution is given by $F(x) = \frac{x^2}{1+x^2}$.

Application to DRM

We directly deduce that

$$\sup_{\overline{q}\in\overline{\mathcal{Q}}_{0,1}} \Phi(\overline{q}) = \int_0^1 \sqrt{\frac{1-u}{u}}\, d\psi(u)$$

Application to DRM

We can retrieve the following classical result:

For $\psi(u) := \mathbf{1}_{u \geq \alpha}$, $\alpha \in (0,1)$, we have

$$\sup_{X \in \mathcal{L}_{\mu,\sigma}} VaR_{\alpha}(X) = \mu + \sigma \sqrt{\frac{1-\alpha}{\alpha}}$$

Application to DRM

We can retrieve the following classical result:

For $\psi(u) := \mathbf{1}_{u \geq \alpha}$, $\alpha \in (0,1)$, we have

$$\sup_{X \in \mathcal{L}_{\mu,\sigma}} VaR_\alpha(X) = \mu + \sigma\sqrt{\frac{1-\alpha}{\alpha}}$$

Free bonus:

$$\inf_{X \in \mathcal{L}_{\mu,\sigma}} VaR_\alpha(X) = \mu - \sigma\sqrt{\frac{\alpha}{1-\alpha}}$$

Application to DRM

Another classical result:

For $\psi(u) := \min(\frac{u}{\alpha}, 1)$, $\alpha \in (0, 1)$, we have

$$\sup_{X \in \mathcal{L}_{\mu,\sigma}} AVaR_{\alpha}(X) = \mu + \sigma\sqrt{\frac{1-\alpha}{\alpha}}$$

More general constraints

Let $u_0, \ldots, u_n$ denote real-valued, continuous functions defined on $\mathbb{R}$. $(u_0, \ldots, u_n)$ form a Tchebycheff system (or a T-system for short) if for any $(t_0, \ldots, t_n)$ with $t_0 < t_1 < \cdots < t_n$, we have $det(A(t_0, \ldots, t_n)) > 0$ where

$$A(t_0, \ldots, t_n) := \begin{pmatrix} u_0(t_0) & u_0(t_1) & \cdots & u_0(t_n) \\ u_1(t_0) & u_1(t_1) & \cdots & u_1(t_n) \\ \vdots & \vdots & \ddots & \vdots \\ u_n(t_0) & u_n(t_1) & \cdots & u_n(t_n) \end{pmatrix}.$$

More general constraints

Let $u_0, \ldots, u_n$ denote real-valued, continuous functions defined on $\mathbb{R}$. $(u_0, \ldots, u_n)$ form a Tchebycheff system (or a T-system for short) if for any $(t_0, \ldots, t_n)$ with $t_0 < t_1 < \cdots < t_n$, we have $det(A(t_0, \ldots, t_n)) > 0$ where

$$A(t_0, \ldots, t_n) := \begin{pmatrix} u_0(t_0) & u_0(t_1) & \cdots & u_0(t_n) \\ u_1(t_0) & u_1(t_1) & \cdots & u_1(t_n) \\ \vdots & \vdots & \ddots & \vdots \\ u_n(t_0) & u_n(t_1) & \cdots & u_n(t_n) \end{pmatrix}.$$

The previous Theorem extends to the case where $\{g_i, i \in I\}$ forms a T-system.

Thank you for your attention