A Variant of Strassen's Theorem with an Application to the Consistency of Option Prices

I. Cetin Gülüm
Joint work with Stefan Gerhold

Vienna University of Technology
3rd Young Researchers Meeting in Probability, Numerics and Finance

June 29, 2016

TECHNISCHE UNIVERSITÄT WIEN

- Let $(M_n)_{n\in\mathbb{N}}$ be a martingale and $\phi:\mathbb{R}\to\mathbb{R}$ convex. Then by Jensen's inequality we have that

$$\mathbb{E}[\phi(M_s)] \leq \mathbb{E}[\phi(M_t)], \quad s \leq t,$$
$$\int_{\mathbb{R}} \phi(x)\, d\mu_s(x) \leq \int_{\mathbb{R}} \phi(x)\, d\mu_t(x), \quad s \leq t.$$

- Let μ_1 and μ_2 be two probability measures on $\mathbb{R}$ with finite mean $(\mathcal{M})$. Then μ_1 is smaller in convex order than μ_2 $(\mu_1 \leq_c \mu_2)$ if

$$\int_{\mathbb{R}} \phi(x)\, d\mu_1(x) \leq \int_{\mathbb{R}} \phi(x)\, d\mu_2(x),$$

for all convex functions $\phi:\mathbb{R}\to\mathbb{R}$.

- Let $(M_n)_{n\in\mathbb{N}}$ be a martingale and $\phi : \mathbb{R} \to \mathbb{R}$ convex. Then by Jensen's inequality we have that

$$\mathbb{E}[\phi(M_s)] \leq \mathbb{E}[\phi(M_t)], \quad s \leq t,$$
$$\int_{\mathbb{R}} \phi(x)\, d\mu_s(x) \leq \int_{\mathbb{R}} \phi(x)\, d\mu_t(x), \quad s \leq t.$$

- Let μ_1 and μ_2 be two probability measures on $\mathbb{R}$ with finite mean ($\mathcal{M}$). Then μ_1 is smaller in convex order than μ_2 ($\mu_1 \leq_c \mu_2$) if

$$\int_{\mathbb{R}} \phi(x)\, d\mu_1(x) \leq \int_{\mathbb{R}} \phi(x)\, d\mu_2(x),$$

for all convex functions $\phi : \mathbb{R} \to \mathbb{R}$.

Strassen's Theorem

Strassen's Theorem, 1965

Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}$. Then there exists a martingale $(M_n)_{n\in\mathbb{N}}$ such that $M_n \sim \mu_n$ if and only if $\mu_s \leq_c \mu_t$ for all $s \leq t$.

Lemma

Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}$ and define the call function of μ_n as

$$R_{\mu_n}(x) = \int_{\mathbb{R}} (y-x)^+ \, \mu_n(dy), \quad x \in \mathbb{R}.$$

Then $\mu_s \leq_c \mu_t$ for all $s \leq t$ if and only if $(\mu_n)_{n\in\mathbb{N}}$ has constant mean and

$$R_{\mu_s}(x) \leq R_{\mu_t}(x), \quad \text{for all } x \in \mathbb{R}.$$

Strassen's Theorem

Strassen's Theorem, 1965

Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}$. Then there exists a martingale $(M_n)_{n\in\mathbb{N}}$ such that $M_n \sim \mu_n$ if and only if $\mu_s \leq_c \mu_t$ for all $s \leq t$.

Lemma

Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}$ and define the call function of μ_n as

$$R_{\mu_n}(x) = \int_{\mathbb{R}} (y-x)^+ \, \mu_n(dy), \quad x \in \mathbb{R}.$$

Then $\mu_s \leq_c \mu_t$ for all $s \leq t$ if and only if $(\mu_n)_{n\in\mathbb{N}}$ has constant mean and

$$R_{\mu_s}(x) \leq R_{\mu_t}(x), \quad \text{for all } x \in \mathbb{R}.$$

Application - Classical Problem

- Given a finite set of European call option prices $r_{t,i}$, with maturity $t \in \{1, \dots, T\}$ and strike $K_i \in \{K_1, \dots, K_N\}$ and given the price of the underlying asset S_0, when does there exist an arbitrage-free model which generates these prices?
- A model is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a non-negative martingale S such that

$$\mathbb{E}\left[(S_t - K_i)^+\right] = r_{t,i}.$$

Literature:

- Carr and Madan (2005) → necessary and sufficient conditions
- Davis and Hobson (2007) → arbitrage strategies
- Cousot (2007) → positive bid-ask spread on options (but not on the underlying).

Application - Classical Problem

- Given a finite set of European call option prices $r_{t,i}$, with maturity $t \in \{1, \dots, T\}$ and strike $K_i \in \{K_1, \dots, K_N\}$ and given the price of the underlying asset S_0, when does there exist an arbitrage-free model which generates these prices?
- A model is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a non-negative martingale S such that

$$\mathbb{E}\left[(S_t - K_i)^+\right] = r_{t,i}.$$

Literature:

- Carr and Madan (2005) → necessary and sufficient conditions
- Davis and Hobson (2007) → arbitrage strategies
- Cousot (2007) → positive bid-ask spread on options (but not on the underlying).

Application - Classical Problem

- Given a finite set of European call option prices $r_{t,i}$, with maturity $t \in \{1,\dots,T\}$ and strike $K_i \in \{K_1,\dots,K_N\}$ and given the price of the underlying asset S_0, when does there exist an arbitrage-free model which generates these prices?
- A model is a probability space $(\Omega,\mathcal{F},\mathbb{P})$ and a non-negative martingale S such that

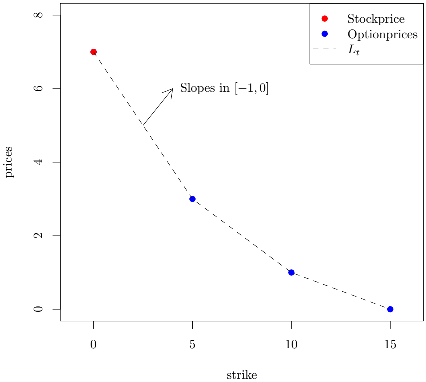
$$\mathbb{E}\left[(S_t - K_i)^+\right] = r_{t,i}.$$

Literature:

- Carr and Madan (2005) → necessary and sufficient conditions
- Davis and Hobson (2007) → arbitrage strategies
- Cousot (2007) → positive bid-ask spread on options (but not on the underlying).

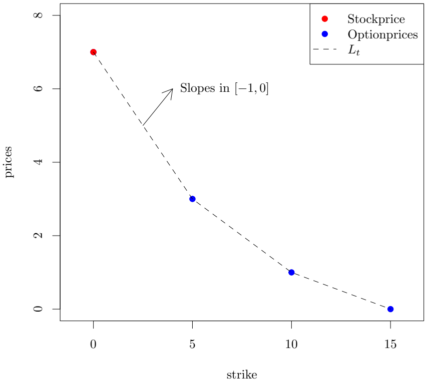
Conditions for single maturities

- For each maturity t the linear interpolation L_t of the points $(K_i, r_{t,i})$ has to be convex, decreasing and all slopes of L_t have to be in $[-1, 0]$.
- Intuition: for every random variable S_t the function $K \mapsto \mathbb{E}[(S_t - K)^+]$ has these properties.



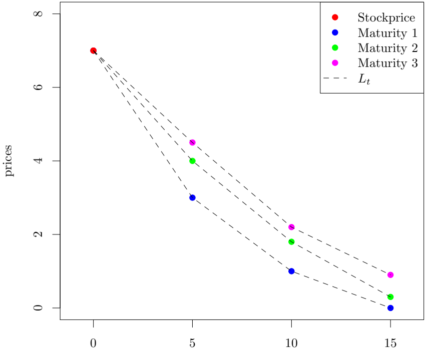
Conditions for single maturities

- For each maturity t the linear interpolation L_t of the points $(K_i, r_{t,i})$ has to be convex, decreasing and all slopes of L_t have to be in $[-1, 0]$.
- Intuition: for every random variable S_t the function $K \mapsto \mathbb{E}[(S_t - K)^+]$ has these properties.



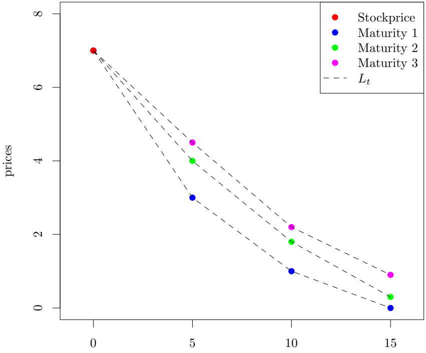
Intertemporal conditions

- For all strikes K_i we have that $r_{t,i} \leq r_{t+1,i}$.
- Intuition: for every martingale $S = (S_t)_{t \in \{0,\dots,T\}}$ the function $t \mapsto \mathbb{E}[(S_t - K)^+]$ is increasing by Strassen's theorem.



Intertemporal conditions

- For all strikes K_i we have that $r_{t,i} \leq r_{t+1,i}$.
- Intuition: for every martingale $S = (S_t)_{t \in \{0,\ldots,T\}}$ the function $t \mapsto \mathbb{E}[(S_t - K)^+]$ is increasing by Strassen's theorem.



Necessary and Sufficient Conditions

- For all maturities t

$$0 \geq \frac{r_{t,i+1} - r_{t,i}}{K_{i+1} - K_i} \geq \frac{r_{t,i} - r_{t,i-1}}{K_i - K_{i-1}} \geq -1, \quad \text{for } i \in \{1, \dots, N-1\},$$

and

$$r_{t,i} = r_{t,i-1} \text{ implies } r_{t,i} = 0, \quad \text{for } i \in \{1, \dots, N\}.$$

- Note that we set $K_0 = 0$ and $r_{t,0} = S_0$ for all $t \in \{1, \dots, T-1\}$.
- For all strikes K_i

$$r_{t,i} \leq r_{t+1,i}, \quad t \in \{1, \dots, T-1\}.$$

- It is possible to state arbitrage strategies if any of these conditions fails.

Necessary and Sufficient Conditions

- For all maturities t

$$0 \geq \frac{r_{t,i+1} - r_{t,i}}{K_{i+1} - K_i} \geq \frac{r_{t,i} - r_{t,i-1}}{K_i - K_{i-1}} \geq -1, \quad \text{for } i \in \{1, \ldots, N-1\},$$

  and

$$r_{t,i} = r_{t,i-1} \text{ implies } r_{t,i} = 0, \quad \text{for } i \in \{1, \ldots, N\}.$$

- Note that we set $K_0 = 0$ and $r_{t,0} = S_0$ for all $t \in \{1, \ldots, T-1\}$.
- For all strikes K_i

$$r_{t,i} \leq r_{t+1,i}, \quad t \in \{1, \ldots, T-1\}.$$

- It is possible to state arbitrage strategies if any of these conditions fails.

Necessary and Sufficient Conditions

- For all maturities t

$$0 \geq \frac{r_{t,i+1} - r_{t,i}}{K_{i+1} - K_i} \geq \frac{r_{t,i} - r_{t,i-1}}{K_i - K_{i-1}} \geq -1, \quad \text{for } i \in \{1, \dots, N-1\},$$

  and

$$r_{t,i} = r_{t,i-1} \text{ implies } r_{t,i} = 0, \quad \text{for } i \in \{1, \dots, N\}.$$

- Note that we set $K_0 = 0$ and $r_{t,0} = S_0$ for all $t \in \{1, \dots, T-1\}$.
- For all strikes K_i

$$r_{t,i} \leq r_{t+1,i}, \quad t \in \{1, \dots, T-1\}.$$

- It is possible to state arbitrage strategies if any of these conditions fails.

Necessary and Sufficient Conditions

- For all maturities t

$$0 \geq \frac{r_{t,i+1} - r_{t,i}}{K_{i+1} - K_i} \geq \frac{r_{t,i} - r_{t,i-1}}{K_i - K_{i-1}} \geq -1, \quad \text{for } i \in \{1, \ldots, N-1\},$$

  and

$$r_{t,i} = r_{t,i-1} \text{ implies } r_{t,i} = 0, \quad \text{for } i \in \{1, \ldots, N\}.$$

- Note that we set $K_0 = 0$ and $r_{t,0} = S_0$ for all $t \in \{1, \ldots, T-1\}$.
- For all strikes K_i

$$r_{t,i} \leq r_{t+1,i}, \quad t \in \{1, \ldots, T-1\}.$$

- It is possible to state arbitrage strategies if any of these conditions fails.

Application - New Problem

- Additional to the classical Problem assume that there is a positive bid-ask spread on the underlying ($\underline{S}_t \leq \overline{S}_t$).
- What is the payoff of a European call option at maturity t?

$$\text{Is it } (\overline{S}_t - K)^+? \quad \text{or } (\underline{S}_t - K)^+?$$

- We assume that there is a third process $(S_t^C)_{t\in\{0,\dots,T\}}$ such that $\underline{S}_t \leq S_t^C \leq \overline{S}_t$ and such that the payoff is given by

$$(S_t^C - K)^+.$$

  Options are cash-settled.
- An arbitrage-free model is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and four non-negative processes:

$$\underline{S}, \overline{S}, S^C, S^*.$$

- S^* is a martingale which evolves in the bid-ask spread: $\underline{S}_t \leq S_t^* \leq \overline{S}_t$.
- S^C is not a traded asset, hence S^C does not have to be a martingale.

Application - New Problem

- Additional to the classical Problem assume that there is a positive bid-ask spread on the underlying ($\underline{S}_t \le \overline{S}_t$).
- What is the payoff of a European call option at maturity t?

$$\text{Is it } (\overline{S}_t - K)^+? \quad \text{or } (\underline{S}_t - K)^+?$$

- We assume that there is a third process $(S_t^C)_{t\in\{0,\dots,T\}}$ such that $\underline{S}_t \le S_t^C \le \overline{S}_t$ and such that the payoff is given by

$$(S_t^C - K)^+.$$

  Options are cash-settled.
- An arbitrage-free model is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and four non-negative processes:

$$\underline{S}, \overline{S}, S^C, S^*.$$

- S^* is a martingale which evolves in the bid-ask spread: $\underline{S}_t \le S_t^* \le \overline{S}_t$.
- S^C is not a traded asset, hence S^C does not have to be a martingale.

Application - New Problem

- Additional to the classical Problem assume that there is a positive bid-ask spread on the underlying ($\underline{S}_t \leq \overline{S}_t$).
- What is the payoff of a European call option at maturity t?

$$\text{Is it } (\overline{S}_t - K)^+? \quad \text{or } (\underline{S}_t - K)^+?$$

- We assume that there is a third process $(S_t^C)_{t\in\{0,\dots,T\}}$ such that $\underline{S}_t \leq S_t^C \leq \overline{S}_t$ and such that the payoff is given by

$$(S_t^C - K)^+.$$

Options are cash-settled.

- An arbitrage-free model is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and four non-negative processes:

$$\underline{S}, \overline{S}, S^C, S^*.$$

- S^* is a martingale which evolves in the bid-ask spread: $\underline{S}_t \leq S_t^* \leq \overline{S}_t$.
- S^C is not a traded asset, hence S^C does not have to be a martingale.

Application - New Problem

- Additional to the classical Problem assume that there is a positive bid-ask spread on the underlying ($\underline{S}_t \leq \overline{S}_t$).
- What is the payoff of a European call option at maturity t?

$$\text{Is it } (\overline{S}_t - K)^+? \quad \text{or } (\underline{S}_t - K)^+?$$

- We assume that there is a third process $(S_t^C)_{t\in\{0,\dots,T\}}$ such that $\underline{S}_t \leq S_t^C \leq \overline{S}_t$ and such that the payoff is given by

$$(S_t^C - K)^+.$$

  Options are cash-settled.
- An arbitrage-free model is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and four non-negative processes:

$$\underline{S}, \overline{S}, S^C, S^*.$$

- S^* is a martingale which evolves in the bid-ask spread: $\underline{S}_t \leq S_t^* \leq \overline{S}_t$.
- S^C is not a traded asset, hence S^C does not have to be a martingale.

Application - New Problem

- Additional to the classical Problem assume that there is a positive bid-ask spread on the underlying ($\underline{S}_t \leq \overline{S}_t$).
- What is the payoff of a European call option at maturity t?

$$\text{Is it } (\overline{S}_t - K)^+? \quad \text{or } (\underline{S}_t - K)^+?$$

- We assume that there is a third process $(S_t^C)_{t\in\{0,\dots,T\}}$ such that $\underline{S}_t \leq S_t^C \leq \overline{S}_t$ and such that the payoff is given by

$$(S_t^C - K)^+.$$

  Options are cash-settled.
- An arbitrage-free model is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and four non-negative processes:

$$\underline{S}, \overline{S}, S^C, S^*.$$

- S^* is a martingale which evolves in the bid-ask spread: $\underline{S}_t \leq S_t^* \leq \overline{S}_t$.
- S^C is not a traded asset, hence S^C does not have to be a martingale.

Application - New Problem

- Additional to the classical Problem assume that there is a positive bid-ask spread on the underlying ($\underline{S}_t \leq \overline{S}_t$).
- What is the payoff of a European call option at maturity t?

$$\text{Is it } (\overline{S}_t - K)^+? \quad \text{or } (\underline{S}_t - K)^+?$$

- We assume that there is a third process $(S_t^C)_{t\in\{0,\dots,T\}}$ such that $\underline{S}_t \leq S_t^C \leq \overline{S}_t$ and such that the payoff is given by

$$(S_t^C - K)^+.$$

  Options are cash-settled.
- An arbitrage-free model is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and four non-negative processes:

$$\underline{S}, \overline{S}, S^C, S^*.$$

- S^* is a martingale which evolves in the bid-ask spread: $\underline{S}_t \leq S_t^* \leq \overline{S}_t$.
- S^C is not a traded asset, hence S^C does not have to be a martingale.

Unbounded Bid-Ask Spread

- If we allow models where the bid ask can get arbitrarily large than there are no intertemporal conditions.
- For all maturities t the following conditions are then necessary and sufficient for the existence of arbitrage-free models:

$$0 \geq \frac{r_{t,i+1} - r_{t,i}}{K_{i+1} - K_i} \geq \frac{r_{t,i} - r_{t,i-1}}{K_i - K_{i-1}} \geq -1, \quad \text{for } i \in \{2, \dots, N-1\},$$

and

$$r_{t,i} = r_{t,i-1} \text{ implies } r_{t,i} = 0, \quad \text{for } i \in \{2, \dots, N\}.$$

- Note that the initial bid and ask price of the underlying $(\underline{S}_0, \overline{S}_0)$ do not appear!

Unbounded Bid-Ask Spread

- If we allow models where the bid ask can get arbitrarily large than there are no intertemporal conditions.
- For all maturities t the following conditions are then necessary and sufficient for the existence of arbitrage-free models:

$$0 \geq \frac{r_{t,i+1} - r_{t,i}}{K_{i+1} - K_i} \geq \frac{r_{t,i} - r_{t,i-1}}{K_i - K_{i-1}} \geq -1, \quad \text{for } i \in \{2, \ldots, N-1\},$$

and

$$r_{t,i} = r_{t,i-1} \text{ implies } r_{t,i} = 0, \quad \text{for } i \in \{2, \ldots, N\}.$$

- Note that the initial bid and ask price of the underlying $(\underline{S}_0, \overline{S}_0)$ do not appear!

Unbounded Bid-Ask Spread

- If we allow models where the bid ask can get arbitrarily large than there are no intertemporal conditions.
- For all maturities t the following conditions are then necessary and sufficient for the existence of arbitrage-free models:

$$0 \geq \frac{r_{t,i+1} - r_{t,i}}{K_{i+1} - K_i} \geq \frac{r_{t,i} - r_{t,i-1}}{K_i - K_{i-1}} \geq -1, \quad \text{for } i \in \{2, \dots, N-1\},$$

and

$$r_{t,i} = r_{t,i-1} \text{ implies } r_{t,i} = 0, \quad \text{for } i \in \{2, \dots, N\}.$$

- Note that the initial bid and ask price of the underlying $(\underline{S}_0, \overline{S}_0)$ do not appear!

Bounded Bid-Ask Spreads

- We focus on models where the bid-ask spread is bounded: there has to exist $\epsilon \geq 0$ and $p \in [0,1]$ such that

$$\mathbb{P}\Big(\overline{S}_t - \underline{S}_t > \epsilon\Big) \leq p.$$

- In particular, $\mathbb{P}\Big(|S_t^C - S_t^*| > \epsilon\Big) \leq p$.
- The option prices allow us to construct measures which correspond to the law of S^C (temporal conditions).
- Strassen's theorem is not applicable anymore since S^C does not have to be a martingale.
  But, S^C has to be *close* to a martingale.

Bounded Bid-Ask Spreads

- We focus on models where the bid-ask spread is bounded: there has to exist $\epsilon \geq 0$ and $p \in [0,1]$ such that

$$\mathbb{P}\left(\overline{S}_t - \underline{S}_t > \epsilon\right) \leq p.$$

- In particular, $\mathbb{P}\left(|S_t^C - S_t^*| > \epsilon\right) \leq p$.
- The option prices allow us to construct measures which correspond to the law of S^C (temporal conditions).
- Strassen's theorem is not applicable anymore since S^C does not have to be a martingale.
  But, S^C has to be *close* to a martingale.

Bounded Bid-Ask Spreads

- We focus on models where the bid-ask spread is bounded: there has to exist $\epsilon \geq 0$ and $p \in [0,1]$ such that

$$\mathbb{P}\Big(\overline{S}_t - \underline{S}_t > \epsilon\Big) \leq p.$$

- In particular, $\mathbb{P}\Big(|S_t^C - S_t^*| > \epsilon\Big) \leq p$.
- The option prices allow us to construct measures which correspond to the law of S^C (temporal conditions).
- Strassen's theorem is not applicable anymore since S^C does not have to be a martingale.
  But, S^C has to be *close* to a martingale.

Bounded Bid-Ask Spreads

- We focus on models where the bid-ask spread is bounded: there has to exist $\epsilon \geq 0$ and $p \in [0,1]$ such that

$$\mathbb{P}\Big(\overline{S}_t - \underline{S}_t > \epsilon\Big) \leq p.$$

- In particular, $\mathbb{P}\Big(|S^C_t - S^*_t| > \epsilon\Big) \leq p$.
- The option prices allow us to construct measures which correspond to the law of S^C (temporal conditions).
- Strassen's theorem is not applicable anymore since S^C does not have to be a martingale.
  But, S^C has to be *close* to a martingale.

Problem Formulation

Let d be a metric on $\mathcal{M}$ and $\epsilon > 0$.

Formulation 1

Given a sequence $(\mu_n)_{n\in\mathbb{N}}$ in $\mathcal{M}$, when does there exist a martingale $(M_n)_{n\in\mathbb{N}}$ such that

$$d(\mu_n, \mathcal{L}M_n) \leq \epsilon, \quad \text{for all } n \in \mathbb{N}?$$

Formulation 2

Given a sequence $(\mu_n)_{n\in\mathbb{N}}$ in $\mathcal{M}$, when does there exist a sequence $(\nu_n)_{n\in\mathbb{N}}$ which is increasing in convex order (peacock) such that

$$d(\mu_n, \nu_n) \leq \epsilon, \quad \text{for all } n \in \mathbb{N}?$$

We want to solve this problem for different d:

- Infinity Wasserstein distance
- Modified Prokhorov distance
- Prokhorov distance, Lévy distance, modified Lévy distance, Stop-Loss distance,...

Problem Formulation

Let d be a metric on $\mathcal{M}$ and $\epsilon > 0$.

Formulation 1

Given a sequence $(\mu_n)_{n\in\mathbb{N}}$ in $\mathcal{M}$, when does there exist a martingale $(M_n)_{n\in\mathbb{N}}$ such that

$$d(\mu_n, \mathcal{L}M_n) \leq \epsilon, \quad \text{for all } n \in \mathbb{N}?$$

Formulation 2

Given a sequence $(\mu_n)_{n\in\mathbb{N}}$ in $\mathcal{M}$, when does there exist a sequence $(\nu_n)_{n\in\mathbb{N}}$ which is increasing in convex order (peacock) such that

$$d(\mu_n, \nu_n) \leq \epsilon, \quad \text{for all } n \in \mathbb{N}?$$

We want to solve this problem for different d:

- Infinity Wasserstein distance
- Modified Prokhorov distance
- Prokhorov distance, Lévy distance, modified Lévy distance, Stop-Loss distance,...

Problem Formulation

Let d be a metric on $\mathcal{M}$ and $\epsilon > 0$.

Formulation 1

Given a sequence $(\mu_n)_{n\in\mathbb{N}}$ in $\mathcal{M}$, when does there exist a martingale $(M_n)_{n\in\mathbb{N}}$ such that

$$d(\mu_n, \mathcal{L}M_n) \le \epsilon, \quad \text{for all } n \in \mathbb{N}?$$

Formulation 2

Given a sequence $(\mu_n)_{n\in\mathbb{N}}$ in $\mathcal{M}$, when does there exist a sequence $(\nu_n)_{n\in\mathbb{N}}$ which is increasing in convex order (peacock) such that

$$d(\mu_n, \nu_n) \le \epsilon, \quad \text{for all } n \in \mathbb{N}?$$

We want to solve this problem for different d:

- Infinity Wasserstein distance
- Modified Prokhorov distance
- Prokhorov distance, Lévy distance, modified Lévy distance, Stop-Loss distance,...

Infinity Wasserstein distance

- The modified Prokhorov distance with parameter $p \in [0,1]$ is the mapping $d_p^{\mathrm{P}} : \mathcal{M} \times \mathcal{M} \to [0,\infty]$, defined by

$$d_p^{\mathrm{P}}(\mu,\nu) := \inf\Big\{h > 0 : \nu(A) \leq \mu(A^h) + p, \text{ for all closed sets } A \subseteq \mathbb{R}\Big\}$$

where $A^h = \left\{x \in S : \inf_{a \in A} |x - a| \leq h\right\}$.

- The modified Prokhorov distance is not a metric in general!
- The infinity Wasserstein distance W^∞ is defined by

$$W^\infty(\mu,\nu) = d_0^{\mathrm{P}}(\mu,\nu).$$

Infinity Wasserstein distance

- The modified Prokhorov distance with parameter $p \in [0,1]$ is the mapping $d_p^{\mathrm{P}} : \mathcal{M} \times \mathcal{M} \to [0, \infty]$, defined by

$$d_p^{\mathrm{P}}(\mu,\nu) := \inf\Big\{h > 0 : \nu(A) \le \mu(A^h) + p, \text{ for all closed sets } A \subseteq \mathbb{R}\Big\}$$

  where $A^h = \left\{x \in S : \inf_{a \in A} |x - a| \le h\right\}$.
- The modified Prokhorov distance is not a metric in general!
- The infinity Wasserstein distance W^∞ is defined by

$$W^\infty(\mu,\nu) = d_0^{\mathrm{P}}(\mu,\nu).$$

Infinity Wasserstein distance

- The modified Prokhorov distance with parameter $p \in [0,1]$ is the mapping $d_p^{\mathrm{P}} : \mathcal{M} \times \mathcal{M} \to [0,\infty]$, defined by

$$d_p^{\mathrm{P}}(\mu,\nu) := \inf\Big\{ h > 0 : \nu(A) \le \mu(A^h) + p, \text{ for all closed sets } A \subseteq \mathbb{R} \Big\}$$

  where $A^h = \big\{ x \in S : \inf_{a \in A} |x - a| \le h \big\}$.
- The modified Prokhorov distance is not a metric in general!
- The infinity Wasserstein distance W^∞ is defined by

$$W^\infty(\mu,\nu) = d_0^{\mathrm{P}}(\mu,\nu).$$

Minimal distance coupling

Theorem (Strassen 1965, Dudley 1968)

Given measures μ, ν on $\mathbb{R}$, $p \in [0,1]$, and $\epsilon > 0$ there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with random variables $X \sim \mu$ and $Y \sim \nu$ such that

$$\mathbb{P}\Big(\big|X - Y\big| > \epsilon\Big) \le p,$$

if and only if

$$d_p^{\mathrm{P}}(\mu, \nu) \le \epsilon.$$

This is exactly what we need: we are interested in models where

$$\mathbb{P}\Big(|S_t^C - S_t^*| > \epsilon\Big) \le p.$$

Minimal distance coupling

Theorem (Strassen 1965, Dudley 1968)

Given measures μ, ν on $\mathbb{R}$, $p \in [0,1]$, and $\epsilon > 0$ there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with random variables $X \sim \mu$ and $Y \sim \nu$ such that

$$\mathbb{P}\big(|X - Y| > \epsilon\big) \leq p,$$

if and only if

$$d_p^{\mathrm{P}}(\mu, \nu) \leq \epsilon.$$

This is exactly what we need: we are interested in models where

$$\mathbb{P}\big(|S_t^C - S_t^*| > \epsilon\big) \leq p.$$

First answer

Question

Given a sequence $(\mu_n)_{n\in\mathbb{N}}$ in $\mathcal{M}$, $p \in [0,1]$ and $\epsilon > 0$ when does there exist a peacock $(\nu_n)_{n\in\mathbb{N}}$ such that

$$d_p^{\mathrm{P}}(\mu_n, \nu_n) \leq \epsilon, \quad \text{for all } n \in \mathbb{N}?$$

- Answer: If $p > 0$, then there always exists such a peacock!
- **Conclusion:** if we allow models where $\mathbb{P}\left(\overline{S}_t - \underline{S}_t > \epsilon\right) \leq p$, for $p \in (0,1]$. Then for all maturities t the following conditions are necessary and sufficient for the existence of arbitrage-free models:

$$0 \geq \frac{r_{t,i+1} - r_{t,i}}{K_{i+1} - K_i} \geq \frac{r_{t,i} - r_{t,i-1}}{K_i - K_{i-1}} \geq -1, \quad \text{for } i \in \{2, \dots, N-1\},$$

and

$$r_{t,i} = r_{t,i-1} \text{ implies } r_{t,i} = 0, \quad \text{for } i \in \{2, \dots, N\}.$$

First answer

Question

Given a sequence $(\mu_n)_{n\in\mathbb{N}}$ in $\mathcal{M}$, $p \in [0,1]$ and $\epsilon > 0$ when does there exist a peacock $(\nu_n)_{n\in\mathbb{N}}$ such that

$$d_p^{\mathrm{P}}(\mu_n, \nu_n) \leq \epsilon, \quad \text{for all } n \in \mathbb{N}?$$

- Answer: If $p > 0$, then there always exists such a peacock!
- **Conclusion:** if we allow models where $\mathbb{P}\left(\overline{S}_t - \underline{S}_t > \epsilon\right) \leq p$, for $p \in (0,1]$. Then for all maturities t the following conditions are necessary and sufficient for the existence of arbitrage-free models:

$$0 \geq \frac{r_{t,i+1} - r_{t,i}}{K_{i+1} - K_i} \geq \frac{r_{t,i} - r_{t,i-1}}{K_i - K_{i-1}} \geq -1, \quad \text{for } i \in \{2, \ldots, N-1\},$$

and

$$r_{t,i} = r_{t,i-1} \text{ implies } r_{t,i} = 0, \quad \text{for } i \in \{2, \ldots, N\}.$$

First answer

Question

Given a sequence $(\mu_n)_{n\in\mathbb{N}}$ in $\mathcal{M}$, $p \in [0,1]$ and $\epsilon > 0$ when does there exist a peacock $(\nu_n)_{n\in\mathbb{N}}$ such that

$$d_p^{\mathrm{P}}(\mu_n, \nu_n) \leq \epsilon, \quad \text{for all } n \in \mathbb{N}?$$

- Answer: If $p > 0$, then there always exists such a peacock!
- **Conclusion:** if we allow models where $\mathbb{P}\big(\overline{S}_t - \underline{S}_t > \epsilon\big) \leq p$, for $p \in (0,1]$. Then for all maturities t the following conditions are necessary and sufficient for the existence of arbitrage-free models:

$$0 \geq \frac{r_{t,i+1} - r_{t,i}}{K_{i+1} - K_i} \geq \frac{r_{t,i} - r_{t,i-1}}{K_i - K_{i-1}} \geq -1, \quad \text{for } i \in \{2, \dots, N-1\},$$

and

$$r_{t,i} = r_{t,i-1} \text{ implies } r_{t,i} = 0, \quad \text{for } i \in \{2, \dots, N\}.$$

Solution for W^∞ ($p = 0$), Part 1

- Let $B^\infty(\mu, \epsilon)$ be the closed ball wrt. W^∞ with center μ and radius ϵ. Let $\mathcal{M}_m$ be the set of all probability measures on $\mathbb{R}$ with mean $m \in \mathbb{R}$.
- Given $\epsilon > 0$, a measure $\mu \in \mathcal{M}$ and $m \in \mathbb{R}$ such that $B^\infty(\mu, \epsilon) \cap \mathcal{M}_m \neq \emptyset$ there exist unique measures $S(\mu), T(\mu) \in B^\infty(\mu, \epsilon) \cap \mathcal{M}_m$ such that

$$S(\mu) \leq_c \nu \leq_c T(\mu) \quad \text{for all } \nu \in B^\infty(\mu, \epsilon) \cap \mathcal{M}_m.$$

- The call functions of $S(\mu)$ and $T(\mu)$ are given by

$$R_\mu^{\min}(x; m, \epsilon) = R_{S(\mu)}(x) = \Big(m + R_\mu(x - \epsilon) - \big(\mathbb{E}\mu + \epsilon\big)\Big) \vee R_\mu(x + \epsilon),$$

$$R_\mu^{\max}(x; m, \epsilon) = R_{T(\mu)}(x) = \operatorname{conv}\Big(m + R_\mu(\cdot + \epsilon) - \big(\mathbb{E}\mu - \epsilon\big), R_\mu(\cdot - \epsilon)\Big)(x).$$

Solution for W^∞ ($p = 0$), Part 1

- Let $B^\infty(\mu,\epsilon)$ be the closed ball wrt. W^∞ with center μ and radius ϵ. Let $\mathcal{M}_m$ be the set of all probability measures on $\mathbb{R}$ with mean $m \in \mathbb{R}$.
- Given $\epsilon > 0$, a measure $\mu \in \mathcal{M}$ and $m \in \mathbb{R}$ such that $B^\infty(\mu,\epsilon) \cap \mathcal{M}_m \neq \varnothing$ there exist unique measures $S(\mu), T(\mu) \in B^\infty(\mu,\epsilon) \cap \mathcal{M}_m$ such that

$$S(\mu) \leq_c \nu \leq_c T(\mu) \quad \text{for all } \nu \in B^\infty(\mu,\epsilon) \cap \mathcal{M}_m.$$

- The call functions of $S(\mu)$ and $T(\mu)$ are given by

$$R^{\min}_\mu(x;m,\epsilon) = R_{S(\mu)}(x) = \Big(m + R_\mu(x-\epsilon) - \big(\mathbb{E}\mu + \epsilon\big)\Big) \vee R_\mu(x+\epsilon),$$

$$R^{\max}_\mu(x;m,\epsilon) = R_{T(\mu)}(x) = \operatorname{conv}\Big(m + R_\mu(\cdot+\epsilon) - \big(\mathbb{E}\mu - \epsilon\big), R_\mu(\cdot-\epsilon)\Big)(x).$$

Solution for W^∞ $(p = 0)$, Part 1

- Let $B^\infty(\mu, \epsilon)$ be the closed ball wrt. W^∞ with center μ and radius ϵ. Let $\mathcal{M}_m$ be the set of all probability measures on $\mathbb{R}$ with mean $m \in \mathbb{R}$.
- Given $\epsilon > 0$, a measure $\mu \in \mathcal{M}$ and $m \in \mathbb{R}$ such that $B^\infty(\mu, \epsilon) \cap \mathcal{M}_m \neq \varnothing$ there exist unique measures $S(\mu), T(\mu) \in B^\infty(\mu, \epsilon) \cap \mathcal{M}_m$ such that

$$S(\mu) \leq_{\mathrm{c}} \nu \leq_{\mathrm{c}} T(\mu) \quad \text{for all } \nu \in B^\infty(\mu, \epsilon) \cap \mathcal{M}_m.$$

- The call functions of $S(\mu)$ and $T(\mu)$ are given by

$$R_\mu^{\min}(x; m, \epsilon) = R_{S(\mu)}(x) = \Big(m + R_\mu(x - \epsilon) - \big(\mathbb{E}\mu + \epsilon\big)\Big) \vee R_\mu(x + \epsilon),$$

$$R_\mu^{\max}(x; m, \epsilon) = R_{T(\mu)}(x) = \operatorname{conv}\Big(m + R_\mu(\cdot + \epsilon) - \big(\mathbb{E}\mu - \epsilon\big), R_\mu(\cdot - \epsilon)\Big)(x).$$

Solution for W^∞ ($p = 0$), Part 2

Question

Given a sequence $(\mu_n)_{n\in\mathbb{N}}$ in $\mathcal{M}$ and $\epsilon > 0$ when does there exist a peacock $(\nu_n)_{n\in\mathbb{N}}$ such that

$$W^\infty(\mu_n, \nu_n) = d_0^{\mathrm{P}}(\mu, \nu) \le \epsilon, \quad \text{for all } n \in \mathbb{N}?$$

Answer: if and only if

$$I := \bigcap_{n\in\mathbb{N}} [\mathbb{E}\mu_n - \epsilon, \mathbb{E}\mu_n + \epsilon] \ne \varnothing,$$

and there exists $m \in I$ such that for all $N \in \mathbb{N}$, $x_1, \ldots, x_N \in \mathbb{R}$, we have

$$R_{\mu_1}^{\min}(x_1; m, \epsilon) + \sum_{n=2}^{N} \Big(R_{\mu_n}(x_n + \epsilon\sigma_n) - R_{\mu_n}(x_{n-1} + \epsilon\sigma_n) \Big) \le R_{\mu_{N+1}}^{\max}(x_N; m, \epsilon),$$

where $\sigma_n = \operatorname{sgn}(x_{n-1} - x_n)$.
If $\epsilon = 0$ this simplifies to

$$R_{\mu_1}(x) \le R_{\mu_2}(x) \le \cdots \le R_{\mu_{N+1}}(x) \le \ldots.$$

Necessary and Sufficient Conditions for single maturities

- If we restrict ourselves to models where $\mathbb{P}\left(\overline{S}_t - \underline{S}_t > \epsilon\right) = 0$ we get the following temporal conditions:

$$0 \geq \frac{r_{t,i+1} - r_{t,i}}{K_{i+1} - K_i} \geq \frac{r_{t,i} - r_{t,i-1}}{K_i - K_{i-1}} \geq -1, \quad \text{for } i \in \{2, \dots, N-1\},$$

  and

$$r_{t,i} = r_{t,i-1} \text{ implies } r_{t,i} = 0, \quad \text{for } i \in \{2, \dots, N\}.$$

-

$$\frac{r_{t,2} - r_{t,1}}{K_2 - K_1} \geq \frac{r_{t,1} - \overline{S}_0}{K_1 - \epsilon} \quad \text{and} \quad \frac{r_{t,1} - \underline{S}_0}{K_1 + \epsilon} \geq -1.$$

Necessary and Sufficient Conditions for single maturities

- If we restrict ourselves to models where $\mathbb{P}\big(\overline{S}_t - \underline{S}_t > \epsilon\big) = 0$ we get the following temporal conditions:

$$0 \geq \frac{r_{t,i+1} - r_{t,i}}{K_{i+1} - K_i} \geq \frac{r_{t,i} - r_{t,i-1}}{K_i - K_{i-1}} \geq -1, \quad \text{for } i \in \{2, \dots, N-1\},$$

and

$$r_{t,i} = r_{t,i-1} \text{ implies } r_{t,i} = 0, \quad \text{for } i \in \{2, \dots, N\}.$$

- $$\frac{r_{t,2} - r_{t,1}}{K_2 - K_1} \geq \frac{r_{t,1} - \overline{S}_0}{K_1 - \epsilon} \quad \text{and} \quad \frac{r_{t,1} - \underline{S}_0}{K_1 + \epsilon} \geq -1.$$

Necessary Conditions for multiple maturities

- If we restrict ourselves to models where $\mathbb{P}\left(\overline{S}_t - \underline{S}_t > \epsilon\right) = 0$ then we get the following intertemporal conditions:
- If $K_i + \epsilon < K_j - \epsilon\sigma_s < K_l + \epsilon$, $s \leq t$ and $s \leq u$ then the following conditions are necessary:

$$\frac{r_s^{CVB}(\sigma_s, K_j) - r_{t,i}}{\left(K_j - \epsilon\sigma_s\right) - \left(K_i + \epsilon\right)} \leq \frac{r_{u,l} - r_s^{CVB}(\sigma_s, K_j)}{K_l + \epsilon - \left(K_s - \epsilon\sigma_s\right)},$$

$$\frac{r_s^{CVB}(\sigma_s, K_j) - r_{t,i}}{\left(K_j - \epsilon\sigma_s\right) - \left(K_i + \epsilon\right)} \leq 0, \quad \text{and}$$

$$\frac{r_{u,l} - r_s^{CVB}(\sigma_s, K_j)}{K_l + \epsilon - \left(K_s - \epsilon\sigma_s\right)} \geq -1$$

where

$$r_s^{CVB} = r_{1,j_1} + \sum_{t=2}^{s}\left(r_{t,j_t} - r_{t,i_{t-1}}\right) + 2\epsilon\mathbf{1}_{\{\sigma_1=-1\}}.$$

Necessary Conditions for multiple maturities

- If we restrict ourselves to models where $\mathbb{P}\left(\overline{S}_t - \underline{S}_t > \epsilon\right) = 0$ then we get the following intertemporal conditions:
- If $K_i + \epsilon < K_j - \epsilon\sigma_s < K_l + \epsilon$, $s \le t$ and $s \le u$ then the following conditions are necessary:

$$\frac{r_s^{CVB}(\sigma_s, K_j) - r_{t,i}}{\left(K_j - \epsilon\sigma_s\right) - \left(K_i + \epsilon\right)} \le \frac{r_{u,l} - r_s^{CVB}(\sigma_s, K_j)}{K_l + \epsilon - \left(K_s - \epsilon\sigma_s\right)},$$

$$\frac{r_s^{CVB}(\sigma_s, K_j) - r_{t,i}}{\left(K_j - \epsilon\sigma_s\right) - \left(K_i + \epsilon\right)} \le 0, \quad \text{and}$$

$$\frac{r_{u,l} - r_s^{CVB}(\sigma_s, K_j)}{K_l + \epsilon - \left(K_s - \epsilon\sigma_s\right)} \ge -1$$

where

$$r_s^{CVB} = r_{1,j_1} + \sum_{t=2}^{s}\left(r_{t,j_t} - r_{t,i_{t-1}}\right) + 2\epsilon\mathbf{1}_{\{\sigma_1=-1\}}.$$

Conclusion

- If there are no transaction costs on the underlying then necessary sufficient conditions can be derived from Strassen's theorem.
- If there is no bound on the bid-ask spread on the underlying there are no intertemporal conditions.
- If the bid-ask spread is bounded by a constant we need a generalization of Strassen's theorem.
  It can be used to derive consistency conditions.