Martingale Optimal Transport in Higher Dimension

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Outline

Table of Contents

The Monge optimal transport problem

Originally a soil moving problem for building.

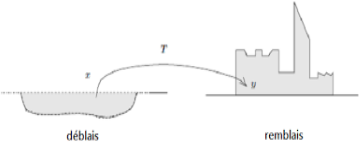

Figure: The Monge problem illustrated.

Figure: The cost of moving building brick.

Probabilistic optimal transport

Kantorovitch has been the one making this problem probabilistic, so that it becomes a linear problem.

- $(\Omega, \mathcal{E}) = (\mathbb{R}^d \times \mathbb{R}^d, \mathcal{B}(\mathbb{R}^d \times \mathbb{R}^d))$
- X and Y the two canonical random variables $\Omega \to \mathbb{R}^d$, $X : (x, y) \mapsto x$ and $Y : (x, y) \mapsto y$.
- $\mathcal{P}(\mu, \nu) := \{\mathbb{P} \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)/\mathbb{P} \circ X^{-1} = \mu, \mathbb{P} \circ Y^{-1} = \nu\}$ the set of all coupling probability laws between μ and ν.

Definition

The optimal transport problem is:

$$\mathbf{P} = \inf_{\mathbb{P} \in \mathcal{P}(\mu,\nu)} \mathbb{E}^{\mathbb{P}}(c(X, Y))$$

The dual problem

We define the dual set:

$$\mathcal{D}_{\mu,\nu}(c) = \Big\{(\phi,\psi) \in \mathrm{L}^1(\mu) \times \mathrm{L}^1(\nu), \\ \forall x,y \in \mathbb{R}^d, c(x,y) \geq \phi(x) + \psi(y)\Big\}$$

Definition

The dual problem is

$$\mathbf{D} := \sup_{(\phi,\psi)\in\mathcal{D}_{\mu,\nu}(c)} \mu(\phi) + \nu(\psi)$$

Remark

If $(\phi,\psi) \in \mathcal{D}_{\mu,\nu}(c)$ *and* $\mathbb{P} \in \mathcal{P}(\mu,\nu)$ *then*

$$\mathbb{E}^{\mathbb{P}}[c(X,Y)] \geq \mathbf{P} \geq \mathbf{D} \geq \mathbb{P}[\phi(X) + \psi(Y)] = \mu(\phi) + \nu(\psi)$$

Kantorovitch Duality

The parallel study of these two problems is justified by the following theorem:

Theorem

If the cost c is lower semicontinuous and dominated by $a \oplus b \in \mathrm{L}(\mu) \oplus \mathrm{L}(\nu)$, then

- *There is duality:* $\mathbf{P} = \mathbf{D}$.
- *There are optimizers* $(\phi^*, \psi^*) \in \mathcal{D}_{\mu,\nu}(c)$ *for* $\mathbf{D}$ *and* $\mathbb{P}^* \in \mathcal{P}(\mu, \nu)$ *for* $\mathbf{P}$.
- *There is a Borel support* $\Gamma \subset \mathbb{R}^{2d}$ *such that* $\mathbb{P} \in \mathcal{P}(\mu, \nu)$ *is concentrated on* Γ *if and only if it is optimal.*

Proof

$$\mathbb{E}^{\mathbb{P}^*}[c(X, Y) - \phi^*(X) - \psi^*(Y)] = \mathbf{P} - \mathbf{D} = 0$$

and $c(X, Y) - \phi^*(X) - \psi^*(Y) \geq 0$. $\square$

Useful cost functions

- $(x, y) \mapsto d(x, y)$ where $d(\cdot, \cdot)$ is a distance.
- $(x, y) \mapsto |x - y|^2$
- $(x, y) \mapsto |x - y|$

Definition

We define the p-Wasserstein distance for $p \geq 1$: for $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$,

$$W^p(\mu, \nu) := \left(\inf_{\mathbb{P} \in \mathcal{P}(\mu,\nu)} \mathbb{E}^{\mathbb{P}}[|X - Y|^p] \right)^{\frac{1}{p}}$$

$(\mathcal{P}(\mathbb{R}^d), W^p)$ is a Polish space.

The Monge-Ampere Equation

How to compute the optimal transport in practice ? Equation on the dual optimizer:

Theorem

- $\mu(dx) = f(x)dx$ *and* $\nu(dy) = g(y)dy$
- $y \mapsto \partial_x c(x, y)$ *is injective*

Then $\mathbb{P}^*[Y = T(X)] = 1$ *with* $T(x) = \partial_x c(x, \cdot)^{-1}(\phi(x))$

$$Det(-D^2\phi(x)+\partial_{xx}c(x, T(x))) = |Det(\partial_{x,y}c(x, T(x)))|\frac{f(x)}{g(T(x))}$$

If $c(x, y) = -x \cdot y$, we get the Monge-Ampere equation:

$$Det(-D^2\phi(x)) = \frac{f(x)}{g(T(x))}$$

Armadillo to ball optimal transport

Example: Optimal transport from Armadillo to ball.

Table of Contents

Change in the Primal and the Dual Problems

Definition

The martingale optimal transport problem and its dual are:

$$\mathbf{P} = \sup_{\mathbb{P}\in\mathcal{M}(\mu,\nu)} \mathbb{E}^{\mathbb{P}}(c(X,Y))$$

and

$$\mathbf{D} := \inf_{(\phi,\psi,h)\in\mathcal{D}_{\mu,\nu}(c)} \mu(\phi)+\nu(\psi)$$

With $\mathcal{M}(\mu,\nu) := \{\mathbb{P}\in\mathcal{P}(\mu,\nu)/\mathbb{E}^{\mathbb{P}}[Y|X]=X \quad a.s.\}$ and

$$\mathcal{D}_{\mu,\nu}(c) = \Big\{(\phi,\psi,h)\in \mathrm{L}^1(\mu)\times\mathrm{L}^1(\nu)\times\mathrm{L}(\mathbb{R}^d,\mathbb{R}^d),$$
$$\forall x,y\in\mathbb{R}^d, c(x,y)\leq\phi(x)+\psi(y)+h(x)\cdot(y-x)\Big\}$$

Application to finance: robust superhedging problem

- S_t is the price vector of d tradable underlyings.
- We are at time 0 and we consider a payoff $c(S_1, S_2)$.
- We can buy vanilla payoffs $\phi(S_1)$ and $\psi(S_2)$.
- Vanilla prices given by the implied measures μ and ν.
- One delta hedge at time 1. Costs $h(S_1) \cdot S_1$ at time 1 and pays $h(S_1) \cdot S_2$ at time 2.
- We want to cover perfectly the payoff with the superhedging.

$$c(S_1, S_2) \leq \phi(S_1) + \psi(S_2) + h(S_1) \cdot (S_2 - S_1)$$

- Problem: cheapest price of the superhedging

$$price(\phi(S_1) + \psi(S_2) + h(S_1) \cdot (S_2 - S_1)) = \mu(\phi) + \nu(\psi)$$

Complete Duality in dimension 1

(Beiglbock-Nutz-Touzi 15) showed that in dimension 1

- Need to consider the $\mathcal{M}(\mu, \nu)$-quasi sure problem.
- Need to use a "convex moderator" to extend $\mu(\phi) + \nu(\psi)$
- There is duality $\mathbf{P} = \mathbf{D}$ for non negative measurable costs.
- The dual problem has an optimizer (ϕ, ψ, h).
- $\mathbb{R}$ is decomposed in "irreducible components" convex and stable by $\mathcal{M}(\mu, \nu)$ on which pointwise duality holds.
- There is a monotone set Γ that supports any optimal probability for the primal problem.

Binary optimal models for specific cost functions

We define for $x \in \mathbb{R}^d$,

$$T(x) := \Gamma_x = \{y \in \mathbb{R}^d / (x, y) \in \Gamma\}$$

(Beiglbock-Juillet 12) or (Henri-Labordere-Touzi 15)

Theorem

We suppose that $d = 1$, $\mu << Leb$ and that $c(x,y) = |x-y|$ or $\partial_{xyy} c > 0$. Then for μ-a.e. $x \in \mathbb{R}$, $Card(T(x)) \leq 2$

This result allows to get two general optimal mappings $T_d : \mathbb{R} \to \mathbb{R}$ and $T_u : \mathbb{R} \to \mathbb{R}$ such that

$$\Gamma = \{(x, T_d(X)), (x, T_u(X)) / x \in \mathbb{R}\}$$

This allows numerical solving.

In higher dimension

(Ghoussoub-Kim-Lim 2015) studied higher dimension.

- $\mathbb{R}^d$ can be split in convex irreducible components for $\mathbb{P}^*$ the optimal probability and the problem could be split on theses components if they had a measurability.
- There are dual optimizers on each components, even if they may not be measurable globally.
- For $c(x,y) = |x-y|$, $\mu << Leb$ and ν atomic, for μ-a.e. $x \in \mathbb{R}^d$, $Card(T(x)) = d+1$.
- They conjecture that more generally, in the case of maximization we can find a.e. $Card(T(x)) = d+1$.

Table of Contents

The generalized integral

Dual problem: minimizing $\mu(\phi)+\nu(\psi)$. But in general we can have $\nu(\psi)=-\mu(\phi)=\infty$. We need to extend the definition of $\mu(\phi)+\nu(\psi)$ using $h^\otimes$ as a compensator:

Definition

We say that $(\phi,\psi)\in \mathrm{L}(\mathbb{R}^d)^2$ is linearly moderated if

$$\sup_{\mathbb{P}\in\mathcal{M}(\mu,\nu)} \mathbb{P}|\phi\oplus\psi+h^\otimes|<\infty$$

for some $h\in \mathrm{L}(\mathbb{R}^d,\mathbb{R}^d)$. Then we define

$$\mu(\phi)\oplus\nu(\psi):=\sup_{\mathbb{P}\in\mathcal{M}(\mu,\nu)} \mathbb{P}[\phi\oplus\psi+h^\otimes]$$

We write the set of these couples of functions $\mathrm{L}(\mu,\nu)$.

(Notation: $\phi\oplus\psi+h^\otimes:=\phi(X)+\psi(Y)+h(X)\cdot(Y-X)$)

Transformation of the dual problem

We need to treat the problem quasi-surely instead of pointwise then.

$$\mathcal{D}^{qs}_{\mu,\nu}(c) := \{(\phi,\psi,h) \in \mathrm{L}(\mu,\nu) \times \mathrm{L}(\mathbb{R}^d,\mathbb{R}^d) / \\ c \le \phi \oplus \psi + h^{\otimes}, \quad \mathcal{M}(\mu,\nu)\text{-q.s.}\}$$

And redefine **D**:

Definition

$$\mathbf{D}(c) := \inf_{(\phi,\psi,h)\in\mathcal{D}^{qs}_{\mu,\nu}(c)} \mu(\phi) \oplus \nu(\psi)$$

Irreducible components

- We get a decomposition of $\mathbb{R}^d$ in irreducible components that depends only on μ and ν.
- We prove that the mapping $x \mapsto C(x)$ is Borel measurable.
- For $A \subset \mathbb{R}^{2d}$,

$$
\begin{aligned}
& (A_\mu \times \mathbb{R}^d) \cap (\mathbb{R}^d \times A_\nu) \cap \{Y \in \overline{C(X)}\} \subset A \\
\Longrightarrow \quad & A \text{ is } \mathcal{M}(\mu,\nu) - q.s. \\
\Longrightarrow \quad & (A_\mu \times \mathbb{R}^d) \cap (\mathbb{R}^d \times A_\nu) \cap \{Y \in C(X)\} \subset A
\end{aligned}
$$

Where A_μ (resp. A_ν) is some μ-a.s. (resp. ν-a.s.) set

- What happens on the border is not clear yet: we have to do assumptions.

Duality theorems

With the assumption we have duality theorems:

- $P = D$ for any nonnegative measurable cost.
- The problem disintegrate in subproblems on the irreducible components where the duality is pointwise.
- There Exists a Borel set Γ such that for any optimal probability $\mathbb{P}^*$,
$$\mathbb{P}^*[\Gamma] = 1$$
- To have a reciprocal property, we would need that for $(\phi, \psi) \in \mathrm{L}(\mu, \nu)$, $\mathbb{P}[\phi \oplus \psi + h^{\otimes}]$ does not depend on $\mathbb{P} \in \mathcal{M}(\mu, \nu)$. (For example when ϕ and ψ are integrable).

Table of Contents

Open questions

- Understanding what happens on the boundary of the components.
- Find whether $\mathbb{P}[\phi \oplus \psi + h^{\otimes}]$ depends on $\mathbb{P} \in \mathcal{M}(\mu, \nu)$ or not.
- How to solve numerically the "martingale" version of the Monge-Ampere equation ?
- Study the behaviour of the problems if we add another constrainst. (Financially speaking, if we add another product in the market).

Questions ?

Figure: An example of Optimal Transport in practice.