Dimension reduction numerical methods for Bermudan options

Scott Sues

Probability, Numerics, and Finance, Le Mans

29 Jun-1 Jul 2016

The University of Queensland
Australia

Outline

1. Dimension reduction approach
2. Fourier cosine method
3. Bermudan option pricing

Abstract

- a dimension reduction approach, built on a combination of Monte Carlo and Fourier Cosine-based methods, for options in high-dimensional models
- realistic jump-diffusion models with stochastic variance and multi-factor stochastic interest rates applicable to foreign exchange options

FX model

- spot price with jumps

$$\frac{\mathrm{d}S(t)}{S(t^-)} = \left(r - \lambda\delta\right)\mathrm{d}t + \sigma\,\mathrm{d}W_S(t) + \mathrm{d}J(t)$$

r : instantaneous interest rate

$W_S(t)$: Wiener process for spot price

$J(t)$: jump process, $J(t) = \sum_{j=0}^{N(t)} (y_j - 1)$

$N(t)$: number of jumps

y : jump amplitude, $\delta = \mathbb{E}\left[y - 1\right]$

λ : jump intensity

FX model

- spot price with jumps

$$\frac{\mathrm{d}S(t)}{S(t^-)} = \big(r - \lambda\delta\big)\,\mathrm{d}t + \sqrt{\nu(t)}\,\mathrm{d}W_S(t) + \mathrm{d}J(t)$$

- stochastic volatility (CIR)

$$\mathrm{d}\nu(t) = \kappa_\nu\big(\bar{\nu} - \nu(t)\big)\,\mathrm{d}t + \sigma_\nu\sqrt{\nu(t)}\,\mathrm{d}W_\nu(t)$$

κ_ν : mean-reversion rate

$\bar{\nu}$: mean volatility

σ_ν : volatility of volatility

$W_\nu(t)$: volatility Wiener process

FX model

- spot price with jumps

$$\frac{\mathrm{d}S(t)}{S(t^-)} = \big(r(t) - \lambda\delta\big)\,\mathrm{d}t + \sqrt{\nu(t)}\,\mathrm{d}W_S(t) + \mathrm{d}J(t)$$

- multi-factor stochastic interest rates

$$r(t) = r(0)\,\mathrm{e}^{-\kappa_r t} + \kappa_r \int_0^t \bar{r}(s)\,\mathrm{e}^{-\kappa_r (t-s)}\,\mathrm{d}s + \sum_{i=1}^{m} X_i(t)$$

$$\mathrm{d}X_i(t) = -\kappa_{r_i} X(t)\,\mathrm{d}t + \sigma_{r_i}\,\mathrm{d}W_{r_i}(t)$$

κ_{r_i} : mean-reversion rate

$\bar{r}(t)$: mean interest rate

σ_{r_i} : volatility

$W_{r_i}(t)$: interest rate Wiener process

FX model

- spot price with jumps

$$\frac{\mathrm{d}S(t)}{S(t^-)} = \big(r(t) - q(t) - \lambda\delta\big)\,\mathrm{d}t + \sqrt{\nu(t)}\,\mathrm{d}W_S(t) + \mathrm{d}J(t)$$

- stochastic volatility

$$\mathrm{d}\nu(t) = \kappa_\nu\big(\bar{\nu} - \nu(t)\big)\,\mathrm{d}t + \sigma_\nu\sqrt{\nu(t)}\,\mathrm{d}W_\nu(t)$$

- multi-factor stochastic interest rates

$$r(t) = \gamma_r(t) + \sum_{i=1}^{m} X_i(t)$$

$$\mathrm{d}X_i(t) = -\kappa_{r_i} X(t)\,\mathrm{d}t + \sigma_{r_i}\,\mathrm{d}W_{r_i}(t)$$

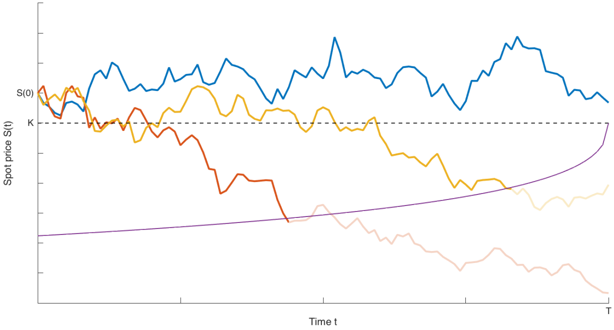
$$q(t) = \gamma_q(t) + \sum_{j=1}^{n} Y_j(t)$$

$$\mathrm{d}Y_j(t) = -\kappa_{q_j} Y(t)\,\mathrm{d}t + \sigma_{q_j}\,\mathrm{d}W_{q_j}(t)$$

Abstract

- Early exercise: Bermudan and American options, barriers
- Put option with exercise payoff $\Phi = (K - S_t)^+ \quad 0 \leq t \leq T$
- Optimal exercise boundary

$$V_0 = \sup_{\tau \in \mathcal{T}} \mathbb{E}\big[D(0,\tau)\,(K - S_\tau)^+\big], \quad D(0,\tau) = \exp\left(-\int_0^\tau r(s)\,\mathrm{d}s\right)$$



Abstract

The approach involves

1. applying conditional Monte Carlo on the variance factor
2. solving the conditional value using the Fourier Cosine method
3. enforcing the optimality condition at each early exercise date using the bundling technique

Numerical results indicate that the approach offers very efficient computation of the prices and hedging parameters.

Outline

Dimension reduction approach

1. Conditional expectation

$$V(S_0,0) = \mathbb{E}\big[D(0,T)\,\Phi(S_T)\big]$$
$$= \mathbb{E}\Big[\,\mathbb{E}\big[D(0,T)\,\Phi(S_T)\,\big|\,\mathcal{G}_T\big]\,\Big]$$

- filtration $\{\mathcal{G}_T, 0 \le t \le T\}$ generated by the processes $\{W_\nu, W_{r_1}, \ldots, W_{r_m}, W_{q_1}, \ldots, W_{q_n}\}$ (all except W_S)

2. Inner expectation: PDE methods
   - Feynman-Kac link between $\mathbb{E}$ and PDE
   - analytical solution using Fourier transforms
   - dimension reduction to 1

3. Outer expectation: Monte Carlo simulation

Conditional PIDE

$$V\big(S(0),0\big) = \mathbb{E}\Big[\mathbb{E}\big[D(0,T)\,\Phi\big(S(T)\big)\;\big|\;\mathcal{G}_T\big]\Big]$$

- Using Feynman-Kac formula

$$V\big(S(0),0\big) = \mathbb{E}\Big[U\Big(S(0),0;\mathcal{G}_T\Big)\Big]$$

where $U(\cdot)$ is the unique solution to the conditional PIDE

$$\frac{\partial U}{\partial t} + \frac{a_{11}^2}{2}\nu(t)S^2\frac{\partial S^2}{\partial^2 U} + \big(r(t) - q(t) - \lambda\delta\big)S\frac{\partial S}{\partial U} - \big(r(t)+\lambda\big)U + \lambda\int_0^\infty U(Sy)\,g(y)\,\mathrm{d}y = 0$$

with terminal condition

$$U\big(S(T),T;\mathcal{G}_T\big) = \Phi\big(S(T)\big)$$

Dimension reduction

- $z(t) = \log S(t)$, $u(z,\cdot) = U(S,\cdot)$, $v(z,\cdot) = V(S,\cdot)$
- Fourier transform $\hat{u}(\xi)$ of conditional PIDE

$$\frac{\partial \hat{u}}{\partial t} + \frac{a_{11}^2}{2}\nu(t)(i\xi)^2\hat{u} + \big(r(t) - q(t) - \lambda\delta - \frac{a_{11}^2}{2}\big)(i\xi)\hat{u} - \big(r(t)+\lambda\big)\hat{u} + \lambda\Gamma(\xi)\hat{u} = 0$$

- ODE can be solved in 1 timestep

$$\hat{v}\,(\xi,t) = \mathbb{E}\Big[\hat{\Phi}(\xi)\exp\Big(-\xi^2\int_0^T \frac{a_{11}^2}{2}\nu(t)\,\mathrm{d}t + i\xi\int_0^T\Big(r(t) - q(t) - \lambda\delta - \frac{\nu(t)}{2}\Big)\,\mathrm{d}t + i\xi\sum_j a_{1j}\int_0^T\sqrt{\nu(t)}\,\mathrm{d}W_j(t) - \int_0^T\big(r(t)+\lambda\big)\,\mathrm{d}t + \lambda T\Gamma(\xi)\Big)\Big]$$

- Dimension reduction, $\mathbb{E}\left[\exp\left(\int_0^T r(t)\,\mathrm{d}t\right)\right]$

$$\hat{v}(\xi,t) = \mathbb{E}\Big[\hat{\Phi}(\xi)\,\exp\big(-G\xi^2 + iF\xi + H + \lambda T\Gamma(\xi)\big)\Big]$$

Dimension reduction

$$G = \frac{a_{1,1}^2}{2} \int_0^T \nu(t)\,\mathrm{d}t + \frac{1}{2} \sum_{k=2}^{h-1} \int_0^T \left(\sum_{j=1}^{m} a_{(j+1),k}\, \beta_{d_j}(t) - \sum_{j=1}^{l} a_{(j+m+1),k}\, \beta_{f_j}(t) + a_{1,k} \sqrt{\nu(t)} \right)^2 \mathrm{d}t$$

$$\begin{aligned} F = & -\frac{1}{2} \int_0^T \nu(t)\,\mathrm{d}t + a_{1,h} \int_0^T \sqrt{\nu(t)}\,\mathrm{d}W_\nu(t) \\ & + \sum_{j=1}^{m} a_{(j+1),h} \int_0^T \beta_{d_j}(t)\,\mathrm{d}W_\nu(t) - \sum_{j=1}^{l} a_{(j+m+1),h} \int_0^T \beta_{f_j}(t)\,\mathrm{d}W_\nu(t) \\ & + \sum_{j=1}^{l} \rho_{s,f_j} \int_0^T \beta_{f_j}(t) \sqrt{\nu(t)}\,\mathrm{d}t - \sum_{k=2}^{h-1} \sum_{j=1}^{m} \int_0^T a_{1,k} a_{(j+1),k}\, \beta_{d_j}(t) \sqrt{\nu(t)}\,\mathrm{d}t \\ & - \sum_{k=2}^{h-1} \int_0^T \sum_{j=1}^{m} a_{(j+1),k}\, \beta_{d_j}(t) \left(\sum_{j=1}^{m} a_{(j+1),k}\, \beta_{d_j}(t) - \sum_{j=1}^{l} a_{(j+m+1),k}\, \beta_{f_j}(t) \right) \mathrm{d}t \\ & + \int_0^T \left(\gamma_d(t) - \gamma_f(t)\right)\,\mathrm{d}t - \lambda\delta T \end{aligned}$$

$$H = -\sum_{j=1}^{m} a_{(j+1),h} \int_0^T \beta_{d_j}(t)\,\mathrm{d}W_\nu(t) - \int_0^T \gamma_d(t)\,\mathrm{d}t + \frac{1}{2} \sum_{k=2}^{h-1} \int_0^T \left(\sum_{j=1}^{m} a_{(j+1),k} \beta_{d_j}(t) \right)^2 \mathrm{d}t - \lambda T$$

European options

$$\hat{v}(\xi,t) = \mathbb{E}\Big[\hat{\Phi}(\xi)\exp\big(-G\xi^2 + iF\xi + H + \lambda T\Gamma(\xi)\big)\Big]$$
$$= \mathbb{E}\Big[\hat{\Phi}(\xi)\,\hat{L}(\xi)\Big]$$

- Inverse transform (convolution theorem)

$$v\big(z(0),0\big) = \mathbb{E}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\phi(x)\,L(z-x)\,\mathrm{d}x\right]$$

- Solution: integrate L, expand $\mathrm{e}^{\lambda T\Gamma(\xi)}$ in Taylor series

$$V(S(0),0) = \mathbb{E}\left[\sum_{n=0}^{\infty}\frac{(\lambda T)^n}{n!}\Big(S(0)\,\mathrm{e}^{F+G+H+W_n}\,\mathcal{N}(d_{1,n}) - K\mathrm{e}^{H}\mathcal{N}(d_{2,n})\Big)\right]$$

Bermudan options

t_n : exercise date timesteps $n = 0, \ldots, N, \quad t_N = T$

$z(t) = \log \frac{S(t)}{K}$

$\phi(z) = [K(1 - \mathrm{e}^z)]^+$, exercise payoff

$v_m(z, t_n)$: value conditional on variance path $m = 1, \ldots, M$

$c_m(z, t_n)$: continuation value

- At maturity, $v_m\big(z, t_N\big) = \phi(z)\,\forall m$
- At prior exercise dates,

$$v_m\big(z, t_n\big) = \max\big(\phi(z),\, c_m\big(z, t_n\big)\big)$$

$$c_m\big(z, t_n\big) = \int_{-\infty}^{+\infty} v_m\big(x, t_{n+1}\big)\, L_m(z, x)\, \mathrm{d}x$$

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2. Fourier cosine method

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Fourier cosine series

If $f(x)$ is an even function periodic on $[-\pi, \pi]$

$$f(x) = \sum_{k=0}^{\infty}{}' A_k \cos(kx)$$

$$A_k = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(kx) \, \mathrm{d}x$$

If $f(x)$ decays rapidly as $x \to \pm\infty$, for a period $[a, b]$:

$$f(x) = \sum_{k=0}^{\infty}{}' A_k \cos\left(\frac{k\pi}{b-a}(x-a)\right)$$

$$A_k = \frac{2}{b-a} \int_a^b f(x) \cos\left(\frac{k\pi}{b-a} x\right) \mathrm{d}x$$

Fourier cosine method

(Fang, Oosterlee, 2008)

Risk-neutral value

$$v(z, t_n) = \mathrm{e}^{-r\Delta t} \int_{\mathbb{R}} v(x, t_{n+1})\, f(x|z)\, \mathrm{d}x$$

Apply Fourier cosine expansion to density $f(x|z)$

$$f(x|z) = {\sum_{k=0}^{\infty}}' A_k(z) \cos\left(\frac{k\pi}{b-a}(x-a)\right)$$

$$\begin{aligned} v(z, t_n) &\approx \mathrm{e}^{-r\Delta t} {\sum_{k=0}^{\infty}}' A_k(z) \left[\int_a^b v(x, t_{n+1}) \cos\left(\frac{k\pi}{b-a}(x-a)\right) \mathrm{d}x\right] \\ &= \mathrm{e}^{-r\Delta t} {\sum_{k=0}^{\infty}}' A_k(z)\, C_k(t_n) \end{aligned}$$

Density coefficient

$$
\begin{aligned}
A_k(z) &= \frac{2}{b-a} \int_a^b f(x|z) \cos\left(\frac{k\pi}{b-a}(x-a)\right) \mathrm{d}x \\
&= \frac{2}{b-a} \operatorname{Re}\left\{ \mathrm{e}^{-ik\pi \frac{a}{b-a}} \int_a^b f(x|z) \exp\left(i \frac{k\pi}{b-a} x\right) \mathrm{d}x \right\} \\
&\approx \frac{2}{b-a} \operatorname{Re}\left\{ \mathrm{e}^{-ik\pi \frac{a}{b-a}} \hat{f}\left(\frac{k\pi}{b-a}; z\right) \right\}
\end{aligned}
$$

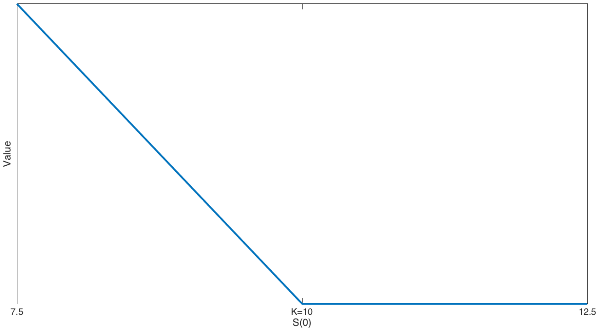
- Integral involving density f replaced with characteristic function $\hat{f}$

Value coefficient

$$C_k(t_n) = \int_a^b v(x, t_{n+1}) \cos\left(\frac{k\pi}{b-a}(x-a)\right) \mathrm{d}x$$

At maturity,

$$C_k(T) = \int_a^b \phi(x) \cos\left(\frac{k\pi}{b-a}(x-a)\right) \mathrm{d}x$$

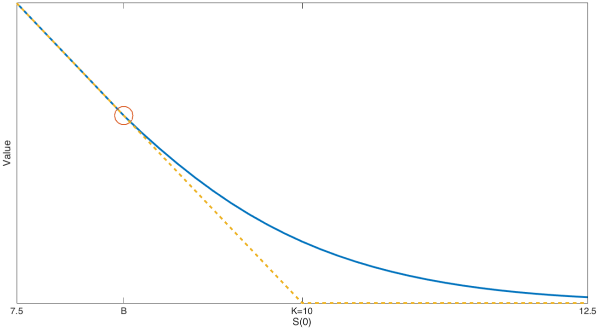


Outline

Value coefficient

At prior exercise dates, separate at optimal exercise barrier $B(t_n)$

$v_m(z, t_n) = \max\left(\phi(z), c_m(z, t_n)\right)$ $\qquad$ $v(z, t_n) = \phi(z)$ for $z \leq B(t_n)$



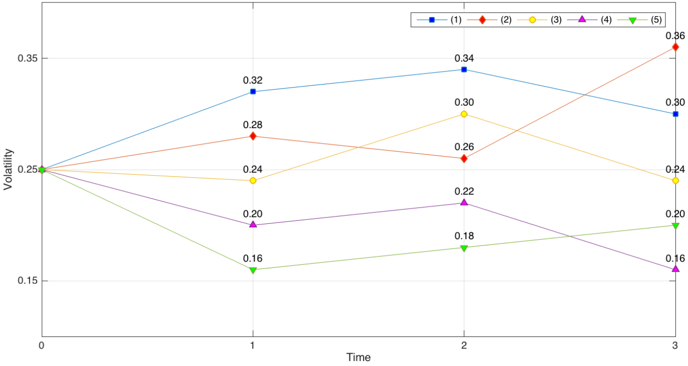
$$C_k(t_n) = \int_a^{B(t_n)} \phi(x) \cos\left(\frac{k\pi}{b-a}(x-a)\right) \mathrm{d}x + \int_{B(t_n)}^{b} v(x, t_{n+1}) \cos\left(\frac{k\pi}{b-a}(x-a)\right) \mathrm{d}x$$

Value coefficient

$$
\begin{aligned}
C_k(t_n) &= \int_a^{B(t_n)} \phi(x) \cos\left(\frac{k\pi}{b-a}(x-a)\right) \mathrm{d}x + \int_{B(t_n)}^{b} v(x, t_{n+1}) \cos\left(\frac{k\pi}{b-a}(x-a)\right) \mathrm{d}x \\
&= I_{\mathsf{PAY},k}(B(t_n)) + \int_{B(t_n)}^{b} v(x, t_{n+1}) \cos\left(\frac{k\pi}{b-a}(x-a)\right) \mathrm{d}x \\
&\approx I_{\mathsf{PAY},k}(B(t_n)) + \\
&\qquad \mathrm{Re}\left\{ \sum_{j=0}^{J} C_j(t_{n+1})\, \hat{f}\left(\frac{k\pi}{b-a}; 0\right) \int_{B(t_n)}^{b} \mathrm{e}^{i\frac{j\pi}{b-a}x} \cos\left(\frac{k\pi}{b-a}(x-a)\right) \mathrm{d}x \right. \\
&= I_{\mathsf{PAY},k}(B(t_n)) + \mathrm{Re}\left\{ \sum_{j=0}^{J} C_j(t_{n+1})\, \hat{f}\left(\frac{k\pi}{b-a}; 0\right) I_{\mathsf{CONT},k}(B(t_n)) \right\}
\end{aligned}
$$

- Solve $C_k(t_n)$ backwards in time using $C_k(t_{n+1})$

Simulation and bundling



Simulation and bundling

$$v(z, t_n) = \sum_k{}' \, \mathrm{e}^{-i\frac{k\pi}{b-a}z} \sum_{m=1}^{M} \frac{L_{k,m}(t_n)\, C_{k,m}(t_n)}{M}$$

$$L_{k,m}(t_n) = \exp\left(-G_m(t_n)\left(\frac{k\pi}{b-a}\right)^2 + i\,(F_m(t_n) - a)\,\frac{k\pi}{b-a} + H_m(t_n)\right)$$

$$C_{k,m}(t_n) = I_{\mathsf{PAY},k}(B_m(t_n)) + \sum_j{}' C_{j,m}(t_{n+1})\,\mathsf{Re}\left\{A_{j,m}(t_n)\, I_{\mathsf{CONT},j}(B_m(t_n))\right\}$$

Simulation and bundling

Price for multiple S and K

$$v(z, t_0) = \sum_k{}' \mathrm{e}^{-i\frac{k\pi}{b-a}z} \sum_{m=1}^{M} \frac{L_{k,m}(t_0)\, C_{k,m}(t_0)}{M}$$

$$\bar{C}_k(t_0) = \sum_{m=1}^{M} \frac{L_{k,m}(t_0)\, C_{k,m}(t_0)}{M}$$

$$V(S(0), K, t_0) = \sum_k{}' \bar{C}_k(t_0) \left(\frac{S(0)}{K}\right)^{-i\frac{k\pi}{b-a}}$$

Algorithm

begin

Simulate M variance paths: $\nu_m(t), \quad m = 1, \dots, M, \quad \nu_m(0) = \nu_0 \,\forall\, m$;
Partition the variance space into P partitions;
$\bar{C}_{k,p}(t_N) \leftarrow I_{\mathsf{PAY},k}(0) \;\forall\, k, p$;

for $t_n \leftarrow t_{N-1}$ **to** t_1 **do**
  for $p \leftarrow 1$ **to** P **do**
    $\mathcal{M} \leftarrow$ index set of variance paths in partition p at time t_n;
    foreach m *in* $\mathcal{M}$ **do**
      Determine early exercise boundary $z \leftarrow B_m(t_n)$;
      Calculate value coefficient $C_{k,m}(t_n) \leftarrow I_{\mathsf{PAY},k}(z) +$
$$\mathrm{Re}\left\{ {\sum_{j=0}^{J}}' \bar{C}_{k,q}(t_{n+1})\, L_{k,m}(t_n)\, I_{\mathsf{CONT},k}(z) \right\}$$
    $\bar{C}_{k,p}(t_n) \leftarrow \sum_{m \in \mathcal{M}} \frac{1}{|\mathcal{M}|} C_{k,m}(t_n)$;

$\bar{C}_k(t_0) \leftarrow \sum_{m=1}^{M} \frac{1}{M} L_{k,m}(t_0)\, C_{k,m}(t_1)$;

$V(S, K) = {\sum_k}' \bar{C}_k(t_0) \left(\frac{S}{K}\right)^{i \frac{k\pi}{b-a}}$;

Thank you!